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WDL TR 1278

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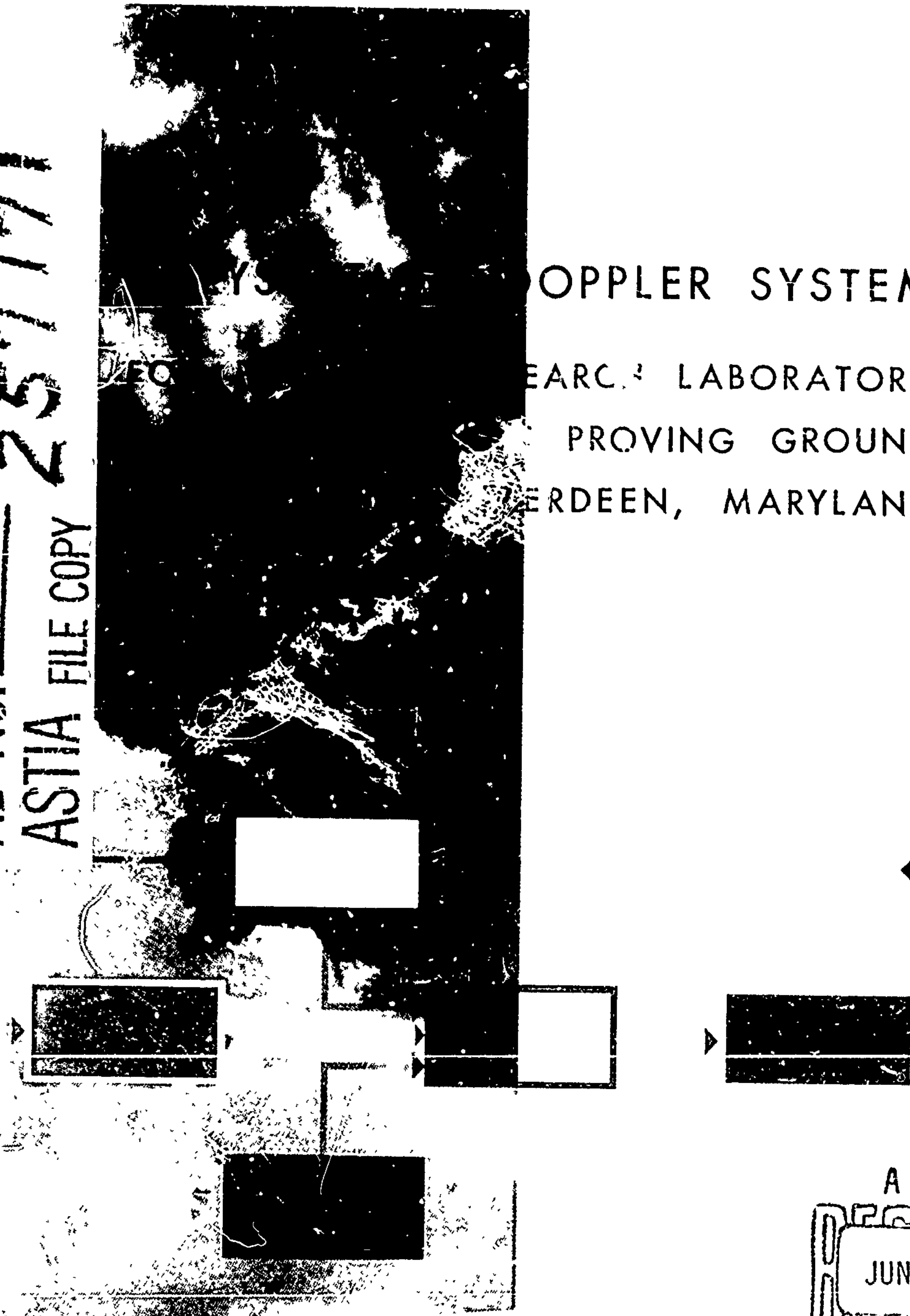

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... PROVING GROUND

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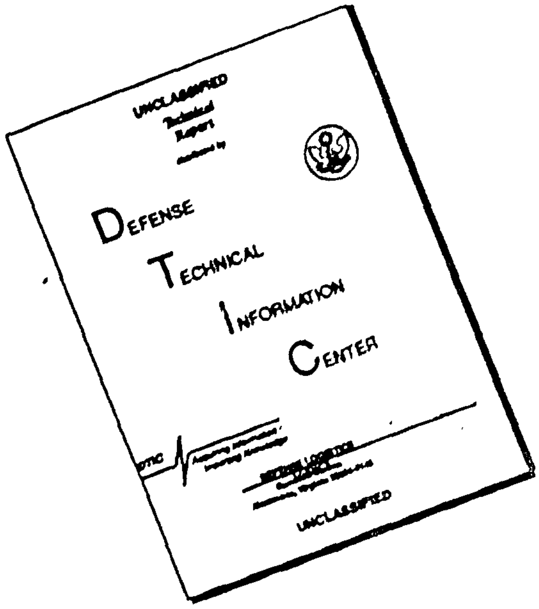

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POLYSTATION DOPPLER SYSTEM

for

BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND
Aberdeen, Maryland
Contract DA-04-200-21X4992-509-ORD-1002

by

PHILCO CORPORATION
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SECTION 1

INTRODUCTION

1.1 Reason for the Study and Statement of Problem Conditions

The Ballistics Research Laboratory (BRL) has developed an iterative method to determine orbital parameters. A need exists to have an initial approximation correct to approximately 10 per cent. This study has been undertaken by the Philco Western Development Laboratories to determine a method for finding initial values. Errors of less than 50 to 75 miles in position, and 1/2 to 1 mile/sec in velocity are specified.

The following conditions were specified for the system:

a. Method: Polystation Doppler

b. Ability: Acquire and track a body to 1000 miles altitude.

c. Limitations:

(1) No knowledge of vehicle position prior to acquisition

(2) No assumption about the nature of the path

(3) No cooperative equipment aboard the vehicle.

1.2 Scope

An error study was performed to determine the dependence of rms error in position and velocity on rms error in the observed quantities. The latter were assumed equal and independent. The error matrix was calculated for a number of circular orbits passing through points in a pre-assigned grid.

A mathematical model was constructed for data reduction.

Position-finding consists of a system of fourteen linear equations solved for approximate values of the position coordinates. These are then adjusted by the standard least square method to their maximum-liklihood values. For technique, see flow chart (Fig. 2a)

For velocity, the interval immediately following the vehicle's passage through the grid plane is considered. For this interval, readings to two stations are used, in conjunction with the adjusted position coordinates, to find approximate values of the horizontal and vertical velocity components. These are then adjusted to their maximum-liklihood values by the use of least squares technique on all four of the readings available for this interval.

Test cases were designed for the mathematical model using circular orbits passing through points in a grid in the fan beam's central plane. The orbit plane inclination was varied through several different values (see Section 4.8 and Fig. 4). For sequence of steps see flow chart, Fig. 3.

SECTION 2

RESULTS AND CONCLUSIONS

The error study made indicates that the BRL conditions can be met by the proposed system. Errors of the order of ten wavelengths can be tolerated at 300 mc The error study which was performed for circular orbits, resulted in data which is presented in the form of graphs, some as curves and some as straight-line graphs

To permit selection of optimum beamwidth, errors were studied for values from 8° to 30° For evaluation of the selected case, charts are presented for constant beamwidths In order to aid in evaluating a system after beamwidth has been decided upon, graphs of error propagation factors were plotted holding beamwidth constant

SECTION 3

RECOMMENDATIONS FOR FUTURE WORK

First in priority for further work would be continuation of the test case computation for the data reduction model. (See Section 6)

Second, as a less major task, a more realistic error calculation for the special case of minimum-time-in-beam orbits should be done

Third, it would be desirable to vary the number of time increments This might decrease the accuracy of the approximate values determined, but would improve the least squares adjustments There is reason to believe a net improvement would result

Fourth, a review of the conditions imposed on the system reveals that the most stringent are embodied in the following two assumptions

a Passage through the beam's mid plane is assumed to occur at mid-time

b Time of entry into and exit from the inner beam are assumed known

In the present system it is the coordinates of this 'piercing point' for which the system is solved Unless the vehicle is so cooperative as to maintain a velocity configuration and reflection characteristics so that it actually does pierce the mid-plane at the mid-time between first and last discernible signals, there is no reason to suppose that this assumption will hold in an operational system Of course, this point is always discernible if a split beam is employed However, in the process of writing the final report, it was determined that both of these conditions could be removed for a fan beam as well and thus permit the solution of the system for any point on the path of the vehicle as it travels through the beam Further, it is possible to commence and cease taking readings at will, rather than having to assume that reading commence simultaneously with entry into the beam and cease simultaneously with exit from the beam Uncertain boundry knowledge of the beam renders this technique questionable For the above reasons, it is strongly suggested that the scope of the present task be extended to include a thorough investigation of the above system.

SECTION 4
SYSTEM DESCRIPTION

4.1 General

The Polystation Doppler System has the ability to acquire and track a moving vehicle. A system is said to acquire if it can determine the position of a body without prior knowledge of its position or velocity. A system is said to track if it can find successive points on a vehicle's path using prior position information. A Doppler Polystation System as used herein refers to an acquisition and tracking system of more than one station which gathers only information regarding diminution or augmentation of received cycles due to motion of the vehicle relative to the station complex.

This particular system must be able to acquire and track any vehicle up to 1000 miles in altitude. The region of coverage is 800 miles long and 1000 miles high, and is limited laterally by the sides of a wedge-shaped beam.

4.2 Configuration

The system is composed of four receivers and one transmitter, located along an arc of a great circle, with the transmitter at one end of the array. All RF signals are transmitted from the ground, reflected from the vehicle and received at four ground stations. No equipment is required aboard the vehicle. A computer to reduce data from the tracking system is the only other equipment required.

4.3 Operation

A coninuous signal of constant frequency is transmitted from a ground station. This signal is reflected from the vehicle and received by the several receivers. The time during which signals are received by all four receivers is the period of observation for collecting data to be reduced for position and velocity information. The distance from the transmitter to the vehicle plus the return to the receiver is defined as the range-sum.

During this interval a certain number of cycles are received. This is compared with the number of cycles transmitted in the same time interval Alternatively, having determined the total time of observation we may divide it into a number of sub-intervals and determine the cycle count difference for these time increments We shall do this in the following way Take the total time of signal reception Divide this into fourteen parts Let the end of the seventh reading be the reference point Calculate the cycle count differences received in intervals of time ending (for the first half) or beginning (for the second half) at this reference time These will be denoted by δ_{1j} They will represent the range-sum differences between the i^{th} time and the reference time measured in wavelengths; that is, $\Delta\delta_{1j}$ is the increase or decrease in the range-sums The difference between the transmitted cycle count and received cycle count of the i^{th} time and the reference time ($t = 0$) is determined by using a common time reference The data is presented to a computer which solves for vehicle position and velocity The time at the center of the total time of observation is determined and position and velocity for that time are determined Knowing this point, position coordinates are easily determined for the other times

4.4 System Synthesis Assumptions and Definitions

The operational requirements of this system do not demand any assumptions about the shape of the earth or its rotation The only requirement is that the five stations be co-planar In synthesizing a system for discussion in this report a spherical earth was assumed for computational convenience in placing the ground stations along a great circle In the study of error as a function of grid position (g-dop), circular orbits were chosen because they were sufficiently general to permit reliable conclusions to be drawn, yet not so complex as to hamper the study or draw attention from the true purpose. The term "cycle count" is defined as the number of doppler cycles counted in some definite time selected for measurement, as opposed to "frequency" which usually indicates the number of cycles in one second

Position is referenced to a right-handed Cartesian coordinate system, rotating with the earth, centered at the transmitter, and oriented as follows. (1) The x-axis passes through the transmitter and the last receiver, positive in the direction of the receiver; (2) the z-axis lies in the plane of the stations and passes through the transmitter, positive "upward" relative to the earth; (3) the y-axis is chosen to complete a right-handed Cartesian system.

Approximate position and velocity data are determined by the solution of certain condition equations. It is assumed that these values sufficiently coincide with the true values to warrant a first order approximation. These unadjusted solutions are used as input to the least squares adjustment technique. This does not mean that this is an iterative technique. One, and only one, adjustment in values is made.

The effects of both special and general relativity were studied. In both cases, the effects were found to be non-existent. Special relativistic effects are cancelled since the transmitter and receivers do not move relative to each other. The possible introduction of an effect due to accelerations of the vehicle relative to the station complex are nullified by assuming that the energy is reflected from the vehicle in zero time.

General relativistic effects are also non-existent. In the proposed system the stations are all located on the surface of a spherical earth. Therefore, all stations are at the same gravitational potential. While this will not be absolutely true for any operational system, nevertheless the anticipated variation in gravity potential at different station locations would be beyond the precision of measurement in the foreseeable future.

4.5 System Synthesis

Having defined the coordinate system, the coordinates of the ground stations may be defined as follows. (1) Successive ground stations are located so that the differences of their x coordinates are equal. (2) Since the ground stations lie along a great circle and the x-axis is defined as a chord intersecting the circle at the transmitter and the most remote receiver, the z-coordinates are determined by knowing the radius of the earth, see Appendix B.4.1 ( ) The y-coordinates are, of course, zero. Thus, viewed as points in the x-z plane, the transmitter is located at (0,0) and the four receivers at $R_1(a,b)$, $R_2(2a,c)$, $R_3(3a,b)$, $R_4(4a,0)$. (See Appendix B.4.1 for derivation of expressions for b,c.) (See Figure 5.) <u>The only condition required by this system is that the five stations be co-planar</u> For computational convenience we assume they are located along a great circle on the surface of a spherical earth.

In our case, the x-coordinate difference referred to in (1) was taken at 200 miles. The resulting values of z were, respectively, for the four receiving stations, approximately 15, 20, 15, 0 miles. Exact figures are derived in Appendix B 4 1. The more accurate figures were used in all computations in order to preserve consistency with an assumed spherical earth of 4000 mile radius.

The direction and beamwidth of transmission and reception create a region of observation which is wedge shaped, with the center plane of symmetry being the xz plane The edge of the outermost wedge is the x-axis, the inner beam edge being tangent to the sphere at receiver three. (See Fig 4) The generating angle of the wedge is the beamwidth.

4 6 <u>Selection of Optimum Beamwidth</u>

To permit the selection of an optimum beamwidth, sigma matrices for position and velocity were calculated for beamwidths of 8°, 10°, 15°, 20°, 25°, and 30°. Charts of $\frac{\sigma_x}{\sigma_M}$, $\frac{\sigma_z}{\sigma_M}$, $\frac{\sigma_{\dot{x}}}{\sigma_M}$, and $\frac{\sigma_{\dot{z}}}{\sigma_M}$ are presented for contours of constant altitudes for various combinations of horizontal displacement x_o and orbital-plane angle. (See Fig 4 and Figs 6 to 14.) σ is the standard deviation and M is the range-sum difference.

4 7 <u>Error Propagation for Fixed Beamdwidth</u>

Having selected a beamwidth, Figs 15 to 20, showing the behavior of $\frac{\sigma_x}{\sigma_M}$, $\frac{\sigma_z}{\sigma_M}$, are presented as functions of altitude, holding constant x_o (horizontal displacement) and $\emptyset$ Figures 21 to 28 are also presented showing the behavior of $\frac{\sigma_x}{\sigma_M}$, $\frac{\sigma_z}{\sigma_M}$, as functions of horizontal displacement for constant altitude and $\emptyset$. These charts are for constant beamwidths and are presented for values of 8° and 15°. By super-imposing on a light table, many of these 8° to 15° pairs of charts may be compared and interpolations performed to find σ values for intermediate values of beamwidth

4.8 Statistical Evaluation of the System

In order to demonstrate the statistical practicability of solving for position and velocity within the error bounds stipulated, a number of representative paths were synthetically generated. These paths pass through the beam at 35 places, intersecting the grid plane (the xz plane) at angles of 0, 15, 30, 45, 60, 75, 90 degrees at each grid point.* In order to synthesize a situation representative of operational conditions, circular orbits were selected which intersected the grid plane at points defined by distances of 50, 200, 350, 500, and 650 statute miles along the x-axis and distances of 150, 300, 450, 600, 750, 900, and 1050 statute miles along the z-axis. (See Fig. 4) The data resulting from this study are presented throughout the text. The equations of the model are included in Appendix B.

* Those for 90°, of course, lie in the grid plane.

SECTION 5

DATA REDUCTION

Approximate position and velocity data are determined by the solution of certain condition equations. It is assumed that these values coincide sufficiently with the true values to warrant a first order approximation. This is equivalent to saying that the value of the variance-covariance matrix does not change appreciably in the region containing both the adjusted and true values.

This does not mean that the mathematical model requires an initial point or prior knowledge of the vehicle's movement It means that the model only requires that the (as yet unadjusted) solutions to the model's equations be sufficiently free from arithmetic error propagation to warrant the first order approximations, for it is these unadjusted solutions that are used as input to the least squares adjustment technique.

It has not been implied that this is an iterative technique. One, and only one, adjustment in values is made. If the unadjusted value is close enough to the correct values to warrant the first order assumptions made, the adjusted value may be shown to be the maximum-likelihood value in a least-squares sense

5 1 Wavelength

In this system wavelength, λ, is assumed to be known, since the transmitter is on the ground.

5 2 Time Duration of Cycle Count

The time of entry is defined as the first point in the beam at which the vehicle is simultaneously observed by all four receivers We shall assign as the time of exit from the beam, a point which is symmetric to the entry point on the opposite side of the grid plane Zero time will be the time of passage through the grid plane, the centermost plane of the beam.

5.3 Cycle Count

Information is received in the form of a plot of doppler frequency in cycles/sec vs time in seconds at each of the four receiver locations for the full period of time (t_{in} to t_{out}) that the vehicle is in the beam of antenna 2 This may then be integrated to yield total cycles counted in any given interval

Alternatively, cycles may be counted directly, thus eliminating several steps. It may be possible to utilize the computer itself as a counter This would be especially attractive since many modern computers could serve as a self-contained time reference while accepting and processing data in real time

5 4 Station Location

It is assumed that the position of the receiver stations (X_j, Z_j) relative to one another and to the transmitter are known to the necessary accuracy. As described earlier (see Section 4, System Description) all results are relative to a coordinate system moving with the stations

5 5 Range-Sum Differences from Cycles Counted

The observed values of range-sum differences will be referred to as M_{ij} or M_k and will be determined as follows:

$$M_{ij} \begin{cases} = \left[c\ (i-8)\Delta t - Q_{ij}\ \lambda\right] & i = 1, \ldots 7 \\ = \left[c\ (i-7)\Delta t - Q_{ij}\ \lambda\right] & i = 8, \ldots 14 \\ & j = 1, \ldots 4 \end{cases}$$

where c is the velocity of propagation in miles per second and λ is the wavelength of the transmitted frequency in miles

When written with a single subscript $M_k = M_{ij}$

for $k = j + 4\ (i-1)$ $j = 1, \ldots 4$

and $k = 1, \ldots, 56$ $i = 1, \ldots 14$

5 6 Position Determination

5 6 1 Range-Sums (Approximate) From Range-Sum Differences

Let ρ_{ij} denote the distance from the transmitter to the i^{th} vehicle position to the j^{th} station; r_{ic}, the distance from the transmitter to the vehicle position; r_{ij}, the distance from the i^{th} vehicle position to the j^{th} receiver; (X_j, Z_j) the coordinates of the j^{th} ground station (note that the transmitter is at the origin); and (x_o, z_o) the coordinates of any point in the grid plane.

$$\rho_{ij} = r_{io} + r_{ij}$$

$$\rho_{ij} = r_{io} - r_{ij}$$

Therefore

$$r_{io}^{2} = x_i^{2} + y_i^{2} + z_i^{2}$$

$$r_{ij}^{2} = (x_i - X_j)^{2} + (y_i)^{2} + (z_i - Z_i)^{2}$$

Let

$$d_j^{2} = X_j^{2} + Z_j^{2}$$

Then

$$(\rho_{ij} - r_{io})^{2} = (r_{ij})^{2}$$

and therefore

$$\rho_{ij}^{2} - 2\, r_{io}\, \rho_{ij} + r_{io}^{2} = r_{io}^{2} - 2\, X_j\, x_i - 2\, Z_j z_i + d_j^{2}$$

and so

$$2\, X_j x_i + 2\, Z_j z_i - 2\, \rho_{ij}\, r_{io} = \rho_{ij}^{2} - d_j^{2}$$

Four equations of this form may be written in determinant form as follows (ρ_{oj} is written as ρ_j for convenience)

$$f_o = f = \begin{vmatrix} \rho_1^{2} & d_1^{2} & \rho_1 & X_1 & Z_1 \\ \rho_2^{2} & d_2^{2} & \rho_2 & X_2 & Z_2 \\ \rho_3^{2} & d_3^{2} & \rho_3 & X_3 & Z_3 \\ \rho_4^{2} & d_4^{2} & \rho_4 & X_4 & Z_4 \end{vmatrix} = 0$$

The determinant is zero since $2z_i$ · column (4) + 2 x_i · column (3) $-2\, r_{io}$ · column (2) = column (1)

Now this equation is valid for any point (x_i, y_i, z_i) in space

When this is written for the i^{th} vehicle position,

$$f_i = \begin{vmatrix} (\rho_1 + M_{i1})^{2} - d_1^{2} & \rho_1 + M_{i1} & X_1 & Z_1 \\ (\rho_2 + M_{i2})^{2} - d_2^{2} & \rho_2 + M_{i2} & X_2 & Z_2 \\ (\rho_3 + M_{i3})^{2} - d_3^{2} & \rho_3 + M_{i3} & X_3 & Z_3 \\ (\rho_4 + M_{i4})^{2} - d_4^{2} & \rho_4 + M_{i4} & X_4 & Z_4 \end{vmatrix} = 0$$

By subtraction, $g_i = f_i - f = 0$ which may be written in the form

$$g_i = \sum_{\rho=1}^{14} \alpha_{i\rho} U_\rho - \beta_i = 0$$

(These will be used as condition equations for the least squares adjustment.)

Where

$U_1 = \rho_1$	$U_5 = \rho_1^2$	$U_9 = \rho_1 \rho_2$	$U_{13} = \rho_2\rho_4$
$U_2 = \rho_2$	$U_6 = \rho_2^2$	$U_{10} = \rho_1 \rho_3$	$U_{14} = \rho_3\rho_4$
$U_3 = \rho_3$	$U_7 = \rho_3^2$	$U_{11} = \rho_1 \rho_4$	
$U_4 = \rho_4$	$U_8 = \rho_4^2$	$U_{12} = \rho_2 \rho_3$	

The solution $U_1, \ldots U_4$ of these fourteen "pseudo" linear equations will be treated as approximations $\rho_1^o, \rho_2^o, \rho_3^o, \rho_4^o$ to the true ρ_j.

5.6.2 Range-Sums to Rectilinear Grid Coordinates

Using three (say ρ_k, ρ_l, ρ_m) out of four of the $\rho_1^o, \ldots \rho_4^o$, the values x_o^o, z_o^o and incidentally r_o^o, may be found by solving

$$2X_k x_o^o + 2Z_k z_o^o - 2\rho_k^o r_o^o = (d_k)^2 - (\rho_k^o)^2$$

$$2X_l x_o^o + 2Z_l z_o^o - 2\rho_l^o r_o^o = (d_l)^2 - (\rho_l^o)^2$$

$$2X_m x_o^o + 2Z_m z_o^o - 2\rho_m^o r_o^o = (d_m)^2 - (\rho_m^o)^2$$

For example

$$x_o^o = \frac{\begin{vmatrix} d_k^2 - (\rho_k^o)^2 & Z_k & \rho_k^o \\ d_l^2 - (\rho_l^o)^2 & Z_l & \rho_l^o \\ d_m^2 - (\rho_m^o)^2 & Z_m & \rho_m^o \end{vmatrix}}{\begin{vmatrix} X_k & Z_k & \rho_k^o \\ X_l & Z_l & \rho_l^o \\ X_m & Z_m & \rho_m^o \end{vmatrix}} = \frac{N}{D}$$

(It is not suggested Cramer's rule be applied in actual cases. Modern matrix inversion techniques almost invariably yield more reliable results.)

The "best" set ρ_k, ρ_1, ρ_m may be determined by evaluating the Jacobian, J, of the transformation for the $C_3^4 = 4$ possible combinations of ρ_1^o ρ_2^o ρ_3^o ρ_4^o

$$J = \begin{vmatrix} \frac{\partial x_o^o}{\partial \rho_k^o} & \frac{\partial z_o^o}{\partial \rho_k^o} & \frac{\partial r_o^o}{\partial \rho_k^o} \\ \frac{\partial x_o^o}{\partial \rho_1^o} & \frac{\partial z_o^o}{\partial \rho_1^o} & \frac{\partial r_o^o}{\partial \rho_1^o} \\ \frac{\partial x_o^o}{\partial \rho_m^o} & \frac{\partial z_o^o}{\partial \rho_m^o} & \frac{\partial r_o^o}{\partial \rho_m^o} \end{vmatrix}$$

For example

$$\frac{\partial x_o^o}{\partial \rho_k^o} = \frac{= N \begin{vmatrix} X_k & Z_k & 1 \\ X_1 & Z_1 & 0 \\ X_m & Z_m & 0 \end{vmatrix} + D \begin{vmatrix} 2\rho_k^o & 0 & .1 \\ d_1^2 - \rho_1^{o2} & Z_1 & \rho_1^o \\ d_m^2 - \rho_m^{o2} & Z_m & \rho_m^o \end{vmatrix}}{-\ D^2}$$

5.6 3 Residuals

Using approximations x_o^o, z_o^o rather than true values will cause the condition equations to yield non-zero values Therefore, we shall define ϵ_i as the residual of the i^{th} condition function

$$g_i(x_o^o, z_o^o) = \sum_{j=1}^{14} \alpha_{ij} \ U_j^o - \beta_i = \epsilon_i \quad i=1, \ldots 14$$

Where

U_j is evaluated at $x = x_o^o$ $z = z_o^o$

5 6.4 Least Squares Adjustment

The observed values M_k (k = 1, .. 56) and the 2 approximate parameters x_o, z_o will be adjusted to satisfy the condition equations while simultaneously minimizing the sum of the squares of the M_k adjustments δ_k; each observation is given equal weight since the errors in the M_k were assumed of equal variance (see Sec. 7.2)

Consider the 14 condition equations, r_i, i=1,.. 14 (see Section 5.6.1)

Let the column vector of the residuals ϵ_i be E

$$E = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_{14} \end{bmatrix} \quad A = \left\| \frac{\partial g_i}{\partial M_k} \right\| ; \quad B = \left\| \frac{\partial g_i}{\partial x_o} \quad \frac{\partial g_i}{\partial z_o} \right\| ; \quad \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_{14} \end{bmatrix}$$

Let

$$\begin{bmatrix} v_{x_o} \\ v_{z_o} \end{bmatrix} = V \text{ be the correction vector of } \begin{bmatrix} x_o \\ x_z \end{bmatrix}$$

The condition equations can be expanded as Taylor series and retaining only the first powers and constant terms, they become

$$h_i = \sum_{k=1}^{56} a_{ik}\, \delta_k + \sum_{q=1}^{2} b_{iq}\, v_q + \epsilon_i = 0$$

or in matrix form

$$A\,\delta + BV + E = 0$$

Define the sum of squares to be minimized as s and subtracting 14 zero terms as follows

$$s = \frac{1}{\sigma_M^2} \sum_{k=1}^{56} \delta_k^2 - 2 \sum_{i=1}^{14} \lambda_i\, h_i$$

where the λ_i are a set of 14 Lagrange multipliers whose column matrix is λ.

To obtain a minimum, differentiate s with respect to δ_k and v_q and equate the resulting expressions to zero.

$$\frac{1}{2} \frac{\partial s}{\partial \delta_k} = \frac{1}{\sigma_M^2} \delta_k - \sum_{i=1}^{14} \lambda_i\, a_{ik}$$

$$\frac{1}{2} \frac{\partial s}{\partial v_q} = \sum_{i=1}^{14} \lambda_i\, b_{iq}$$

Let σ^{o} be a diagonal matrix each element of which has the value σ_M^2

These expressions may be written in matrix form as

$$(\sigma^{o})^{-1} \delta - A^T \lambda = 0$$
$$B^T \lambda = 0$$

then

$$\delta = (\sigma^{o})^{-1} A^T \lambda \text{ and}$$

$$A \delta - BV + E = 0 \text{ becomes}$$

$$A (\sigma^{o})^{-1} A^T \lambda + BV + E = 0 \text{ or}$$

$$\lambda = (A\sigma^{o^{-1}} A^T)^{-1} (BV + E) \text{ and}$$

$$B^T (A\sigma^{o^{-1}} A^T)^{-1} (BV + E) = 0$$

therefore

$$V = (B^T (A\sigma^{o^{-1}} A^T)^{-1} B)^{-1} (B^T (A\sigma^{o^{-1}} A^T)^{-1} E$$

Let

$$H = B^T (A\sigma^{o^{-1}} A^T)^{-1}$$

$$N = H B$$

$(N^{-1}$ may be a desired output)

then

$V = N^{-1} H E$ is determined as the column matrix of the adjustments and the adjusted values of x, z are obtained from

$$\begin{bmatrix} x \\ z \end{bmatrix} = y = y^{o} + V$$

5.7 Velocity Determination

5.7.1 Definitions and Assumptions

Velocity, as used here, is defined as the average velocity in the interval Δt immediately following passage through the beam's central plane

In the determination of velocity, it is assumed that the adjusted values of the position coordinates are available

5 7 2 Condition Equations

Refer to Section 5.6.1 on position finding condition equations for definitions

The four condition equations for velocity are:

$$\frac{M_{ij}}{\Delta t} = x_o \frac{\partial \rho_j}{\partial x_o} + z_o \frac{\partial \rho_j}{\partial z_o}$$

$$j = 1, \quad 4$$

where

$$\frac{\partial \rho_j}{\partial x_o} = \frac{x_o}{r_o} + \frac{x_o \cdot X_j}{r_j}$$

$$\frac{\partial \rho_j}{\partial z_o} = \frac{z_o^o}{r_o} + \frac{z_o^o \quad Z_j}{r_j}$$

As may be seen $\frac{\partial \rho_j}{\partial x_o}$ and $\frac{\partial \rho_j}{\partial z_o}$ are evaluated for the adjusted values x_o, z_o of position

Of these four equations, one particular pairing may yield better results than all others To determine which set of two will yield best results, the Jacobian of the transformation $M_p, M_q \longrightarrow x_o^o, z_o^o$ may be evaluated for each of the $C_2^4 = 6$ cases

$$J = (\Delta t)^2 \begin{vmatrix} \frac{\partial \rho_p}{\partial x_o} & \frac{\partial \rho_p}{\partial z_o} \\ \frac{\rho_q}{x_o} & \frac{\rho_q}{z_o} \end{vmatrix}$$

Where the elements of J are evaluated for $x = x_o$, $z = z_o$

The solution of this system will yield values of velocity components which we view as approximations These will be denoted by $\dot{x}_o^o$ and z_o^o.

5.7.3 Residuals

Define ϵ_{j+14} $j = 1,2,3,4$ as follows.

$$\frac{M_{8j}}{\Delta t} - x_o \frac{\partial \rho_j}{\partial x_o} - z_o \frac{\partial \rho_j}{\partial z_o} = \epsilon_p$$

where $p = j + 14$

Using approximate values $x_o^{\,o}$ and $z_o^{\,o}$ instead of true values will cause the ϵ_{j+14} to be non-zero.

5.7.4 Least Square Adjustment

Let $v_{\dot{x}}$, v_z denote the adjustments to $\dot{x}_o^{\,o}$, $\dot{z}_o^{\,o}$, and $\dot{x}_o$, $\dot{z}_o$ the adjusted values. It is assumed that the adjusted values x_o and z_o used are error-less.

To find the corrections to $\dot{x}_o^{\,o}$ and $\dot{z}_o^{\,o}$ we evaluate

$$S^{-1} = \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{\dot{z}\dot{z}} \end{pmatrix}$$

as in section 7.3 and the matrix

$$\begin{pmatrix} \frac{\partial \rho_1}{\partial x_o} & \cdot & \frac{\partial \rho_4}{\partial x_o} \\ \frac{\partial \rho_1}{\partial z_o} & \cdots & \frac{\partial \rho_4}{\partial z_o} \end{pmatrix}$$

Then

$$\begin{bmatrix} v_x \\ v_{\dot{z}} \end{bmatrix} = (\Delta t)^2 \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{\dot{z}\dot{x}} & \sigma_{\dot{z}\dot{z}} \end{pmatrix} \begin{pmatrix} \frac{\partial \rho_1}{\partial x_o} & \cdots & \frac{\partial \rho_4}{\partial x_o} \\ \frac{\partial \rho_1}{\partial z_o} & \cdots & \frac{\partial \rho_4}{\partial z_o} \end{pmatrix} \begin{pmatrix} \epsilon_{15} \\ \cdot \\ \cdot \\ \epsilon_{18} \end{pmatrix}$$

and so $$\begin{bmatrix} x_o \\ z_o \end{bmatrix} = \begin{bmatrix} \dot{x}_o^{\,o} \\ \dot{z}_o^{\,o} \end{bmatrix} + \begin{bmatrix} v_{\dot{x}_o} \\ v_{\dot{z}_o} \end{bmatrix}$$

SECTION 6
TEST CASE FOR DATA REDUCTION MODEL

6.1 Feasibility Study

In order to show the feasibility of the data reduction method described in section 5 a typical case was selected for simulation on a computer. This work was in progress at the termination of the contract. The following is a description of the proposed test case, (see flow chart, Figure 3). At the time of cessation of work, the matrix involved in position finding had not been inverted. It presently developed that this was due to calculation of erroneous coefficients. A delay caused by the unavailability of the computer and subsequent funds shortages did not permit the simulation of the typical case to be resumed. As mentioned in Section 3, it is hoped that an extension in scope of the present work may permit us to pursue this investigation.

Synthetic data was generated for the minimum beam width of 8° and $x_c=400$, $z_c=450$, $\emptyset=75^{\circ}$, and a circular orbit using the methods of Appendix B. As in an actual case, an approximate position was to have been found, then adjusted. See section 5 for technique.

In the calculation of the position variance-covariance matrix for the least squares adjustment of position in the test case, the approximate values of x_c and z_c are used rather than the true grid point coordinates. This is done to simulate most closely operational conditions. For velocity, we proceed as follows. Using the corrected values of x_c and z_c, $\dot{x}_o^{o}$ and z_o^{o} are found, and these are then corrected using least squares as before. (See section 5)

SECTION 7

ERROR PROPAGATION STUDY

7.1 General Discussion

This section describes the method used in the preparation of the σ/σ curves. (There is, therefore, some duplication between this section and the discussion of least squares in Section 5, Data Reduction.) To permit the selection of an optimum beamwidth, the position error matrix, N^{-1}, and the velocity error matrix, S^{-1}, were calculated for beamwidths of 8^o and $10(5)30^o$.

7.1.1 Results

Charts of $\frac{\sigma_x}{\sigma_M}$, $\frac{\sigma_z}{\sigma_M}$, $\frac{\sigma_{\dot{x}}}{\sigma_M}$, $\frac{\sigma_{\dot{z}}}{\sigma_M}$ vs beamwidth as abcissa are presented for contours of constant altitude, z_o, for various combinations of horizontal displacement x_o and circular orbit inclination (Figures 6 to 14.)

Having selected a beamwidth, line charts showing the behavior of $\frac{\sigma_x}{\sigma_M}$, $\frac{\sigma_z}{\sigma_M}$, are presented as functions of altitude (z_o), holding x_o and Ø constant (see Figures 15 to 20). Charts are also presented showing the behavior of $\frac{\sigma_x}{\sigma_M}$, $\frac{\sigma_z}{\sigma_M}$, $\frac{\sigma_{\dot{x}}}{\sigma_M}$, $\frac{\sigma_{\dot{z}}}{\sigma_M}$, as a function of horizontal displacement (x_o), for constant altitude (z_o), and for orbit plane angle (Ø) (See Appendix C for complete list of Figures.)

7.1.2 Derivation

The variance-covariance matrix among the set of variables $M_1, \ldots M_{56}$ is a symmetrical square array whose elements are the expected values of the cross products of the errors in the M's

$$\sigma_{M_i M_j} = E(\Delta M_i \, \Delta M_j)$$

$$\sigma_{MM^T} = \begin{vmatrix} \sigma_{M_1 M_1} & \cdots & \sigma_{M_1 M_{56}} \\ \vdots & & \vdots \\ \sigma_{M_{56} M_1} & \cdots & \sigma_{M_{56} M_{56}} \end{vmatrix} .$$

$\sigma_{M_iM_i} = E\left[(\Delta M_i)^2\right] = \sigma_{Mi}^{\ 2}$ is the variance. The notation $\sigma_{MM}T$ arises from considering the column matrix of the M_i; thus

$$M\,M^T = \begin{bmatrix} M_1 \\ \vdots \\ M_{56} \end{bmatrix} \begin{bmatrix} M_1 & \cdots\cdots & M_{56} \end{bmatrix} = \begin{bmatrix} M_1M_1 & \cdots\cdots & M_1M_{56} \\ \vdots & & \vdots \\ M_{56}M_1 & \cdots\cdots & M_{56}M_{56} \end{bmatrix}$$

Suppose that y_j ($j = 1,2$)are a set of variables which are functions of M_k

$$y_1 = x_o\ (M_1, \ldots M_{56})$$

$$y_2 = z_o\ (M_1, \ldots M_{56})$$

then

$$\Delta y_j = \sum_{i=1}^{56} \frac{\partial y_j}{\partial M_i}\ \Delta M_i$$

And

$$\Delta y_j\ \Delta y_k = \left(\sum_{i=1}^{M} \frac{\partial y_j}{\partial M_i} \Delta M_i \right) \left(\sum_{q=1}^{M} \frac{\partial y_k}{\partial M_q}\ \Delta M_q \right)$$

$$= \sum_{i=1}^{M} \sum_{q=1}^{M} \frac{\partial y_j}{\partial M_i} \frac{\partial y_k}{\partial M_q} \Delta M_i\, \Delta M_q \text{ and taking}$$

expected values

$$\sigma_{y_jy_k} = \sum_{i=1}^{M} \sum_{q=1}^{M} \frac{\partial y_j}{\partial M_i} \frac{\partial y_k}{\partial M_q}\ \sigma_{M_iM_q}\ ;$$

then

$$\sigma_{YY}T = \left[\frac{\partial y_j}{\partial M_i}\right] \quad \sigma_{MM}T = \left[\frac{\partial y_j}{\partial M_i}\right]^T .$$

Let the x and z coordinates of the grid point be denoted by x_o and z_o, respectively. ALL FUNCTIONS OF x AND z IN THIS SECTION WILL BE TREATED AS THOUGH EVALUATED AT $x = x_o$, $z = z_o$, $\dot{x} = \dot{x}_o$, $\dot{z} = \dot{z}_o$. Note for actual solution (including the test case) : to adjust the approximate values, $x_o^{\ o}$, $z_o^{\ o}$ and $\dot{x}_o^{\ o}$, $\dot{z}_o^{\ o}$, by the use of least squares methods, the variance-covariance matrix is calculated for these approximate values.

The values of the parameters for which σ/σ matrices are calculated as

$$x_o = 50(150)650 \text{ statute miles}$$

$$z_o = 150(150)1050 \text{ statute miles}$$

$$\emptyset = 0(15)90^\circ$$

$$\text{beamwidth} = b\ w = 8^\circ,\ 10(5)30^\circ$$

(For $\emptyset = 0^\circ$, the first run showed values of σ/σ which are probably the square of the true values (1) This is due to the symmetry of this case which causes readings on the two sides of the central plane to be equal This effectively reduces the amount of information by one-half and also diminishes the number of condition equations by one-half. It was intended that this case be re-run with a more irregular scale, but it was not possible to do so in the time remaining.)

Assumptions

1 Covariances $\sigma_{M_q M_p}$ of the observed quantities M_1, M_{56} are zero for $p \neq q$

2 Variances $\sigma_{M_p}^2$ of the observed quantities are equal for all p and will be denoted by σ_M^2.

3 One observation is made for each of the quantities $M_1, .\ M_{56}$.

4 Fourteen condition equations are used for position.

5 Four condition equations are used for velocity.

6 Second order terms in the Taylor expansion of the condition equations (written as residuals) may be ignored

7. The variance-covariance matrix remains approximately constant in r ons containing both the approximate and adjusted values.

(1) This hypothesis is made on the basis of a study of the trend in behavior of σ/σ as $\emptyset$ varies from 75° to 15°. Extrapolation seems to indicate a good fit for $\emptyset = 0$ for the square root of the values for the symmetric case

7.2 Position Derivations

The condition equations are

$$A^{(i)} - A^{(o)} = g_i\ (M_{ij},\ x_o,\ z_o) = \epsilon_i \qquad i = 1, \ldots 14$$

Let $E = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_{14} \end{bmatrix}$

$$A = \left\| a_{ik} \right\| = \left\| \frac{\partial \epsilon_i}{\partial M_k} \right\| \quad i = 1, \ldots 14 \quad k = 1, \ldots 56$$

Note: $a_{ik} \neq 0$ only for $k = j + 4\ (i-1)$

$j = 1, \quad 14 \qquad i = 1, \quad 14$

$$a_{ik} = 2\ (\rho_j + M_{ij})\ A^{(i)}_{j1} + A^{(i)}_{j2}$$

where

$j = k - 4\ (i-1)$ and $A^{(i)}_{jq}$ is the jq^{th} cofactor (signed minor) of the determinant

$$A^{(i)} = \begin{vmatrix} (\rho_1 + M_{i1})^2 - d_1^{\,2} & \rho_1 + M_{i1} & X_1 & Z_1 \\ (\rho_2 + M_{i2})^2 - d_2^{\,2} & \rho_2 + M_{i2} & X_2 & Z_2 \\ (\rho_3 + M_{i3})^2 - d_3^{\,2} & \rho_3 + M_{i3} & X_3 & Z_3 \\ (\rho_4 + M_{i4})^2 - d_4^{\,2} & \rho_4 + M_{i4} & X_4 & Z_4 \end{vmatrix}$$

Let

$$B = \left\| b_{iq} \right\| = \left\| \frac{\partial \epsilon_i}{\partial x} \quad \frac{\partial \epsilon_i}{\partial z} \right\|$$

$$= \left\| \sum_{j=1}^{4} \frac{\partial \epsilon_i}{\partial \rho_j} \frac{\partial \rho_j}{\partial x} \quad \sum_{j=1}^{4} \frac{\partial \epsilon_i}{\partial \rho_j} \frac{\partial \rho_j}{\partial z} \right\|$$

Let

$$C = \left\| c_{ij} \right\| = \left\| \frac{\partial \epsilon_i}{\partial \rho_j} \right\| \qquad \begin{array}{l} i = 1, \ldots 14 \\ j = 1, \ldots 4 \end{array}$$

$$D = \left\| \frac{\partial \rho_j}{\partial x} \quad \frac{\partial \rho_j}{\partial z} \right\|$$

Then

$$c_{ij} = 2\,(\rho_j + M_{ij})\,A_{j1}^{(i)} + A_{j2}^{(i)}$$

$$- 2\,\rho_j\,A_{j1}^{(0)} - A_{j2}^{(0)} ,$$

and

$$D = \left\| d_{jq} \right\| = \left\| d_{j1} \quad d_{j2} \right\|$$

$$= \left\| \frac{x}{r_o} + \frac{x-X_j}{r_j} \quad \frac{z}{r_o} + \frac{z - Z_j}{r_j} \right\|$$

$$j = 1, \ldots 4$$

Note that $(r_o)^2 = (x)^2 + (z)^2$

$$(r_j)^2 = (x - X_j)^2 + (z - Z_j)^2$$

Then $B = CD$

Let σ^o be a diagonal matrix each element of which has the value σ_{MM}. In our case we assume the $\sigma_{MM} = \sigma_M^2$ are all equal and set $= 1$.

Then let $H = B^T\,(A\,\sigma^{o^{-1}}\,A^T)^{-1}$

$$= B^T\,(A\,A^T)^{-1}$$

and let $N = H\,B$

then $N^{-1} = \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{vmatrix}$

7.3 Velocity Derivations

The condition equations are

$$\frac{M_{ij}}{\Delta t} - \dot{x}_o \frac{\partial \rho_j}{\partial x} - \dot{z}_o \frac{\partial \rho_j}{\partial z} = \epsilon_p$$

$$j = 1, \ldots 4 \qquad p = j + 14$$

where

$$M_{8j} = M_k \text{ with } k = 28 + j$$

and where

$\frac{\partial \rho_j}{\partial x}$ and $\frac{\partial \rho_j}{\partial z}$ - are evaluated for the adjusted x_o and z_o.

$$\text{Let } A_v \equiv A_{velocity} = \left\| a_{rj} \right\| = \left\| \frac{\partial \epsilon_p}{\partial M_{8j}} \right\| \qquad \begin{array}{l} r = 1, \dots 4 \\ j = 1, \dots 4 \\ p = r + 14 \end{array}$$

Then

$$a_{rj} = \frac{1}{\Delta t} \quad \text{for } r = j$$

$$= 0 \quad \text{for } r \neq j$$

$$\text{Let } B_v = B_{velocity} \quad \left\| b_{rq} \right\| = \left\| \frac{\partial \rho_p}{\partial \dot{x}} \quad \frac{\partial \rho_p}{\partial \dot{z}} \right\|$$

$$\text{Then } b_{p1} = - \frac{\partial \rho_p}{\partial x} \qquad\qquad \begin{array}{l} r = 1, \dots 4 \\ q = 1, 2 \\ p = j + 14 \end{array}$$

$$b_{p2} = - \frac{\partial \rho_p}{\partial z}$$

$$S^{-1} = \left[B_v^T \; (A_v A_v^T)^{-1} \; B_v \right]^{-1}$$

$$= \left\| \begin{matrix} \sigma_{\dot{x}\dot{x}} & \sigma_{\dot{x}\dot{z}} \\ \sigma_{\dot{z}\dot{x}} & \sigma_{\dot{z}\dot{z}} \end{matrix} \right\|$$

The curves for $\frac{\sigma_{\dot{x}}}{\sigma_M}$ and $\frac{\sigma_{\dot{z}}}{\sigma_M}$ are straight lines on log log paper, (see Figures 6abcd to 14abcd), when plotted vs beamwidth, and so $\frac{\sigma_{\dot{x}}}{\sigma_M}$, $\frac{\sigma_{\dot{z}}}{\sigma_M} = k\,(b.w.)^n$. It turns out that in all cases $n = -1$ and the value of k is 10 times the sigma ratio evaluated for b.w. = 10^o.

APPENDIX A

NOTATION

APPENDIX A

NOTATION

x	horizontal dimension coordinate
y	transverse horizontal dimension coordinate
z	vertical dimension coordinate
x_o	x coordinate of grid point
z_o	z coordinate of grid point
t	time parameter
Δt	duration of single time increment Δt
i	vehicle path point index (reading index)
j	station index
g_i	condition equation functions
A	least squares matrix
$E(\ )$	expected value of ( )
E_i	residuals
M_{ij}, M_k, Δ_{ij} or Δ_k	range sum differences corresponding to the j^{th} reading at the i^{th} station, $k=4(i-1)+j$
$R_1, \ldots, R_4$	receivers
T	transmitter
$\emptyset$	orbital plane inclination angle
ψ	beamwidth
ρ_{ij}	range-sum, i e distance transmitter-vehicle-ground.
σ_{xx}, σ_x^2	variance
σ_{xz}	covariance
ρ_j	range sums to grid point
x_i	x coordinate of the i^{th} path-point
y_i	y coordinate of the i^{th} path-point
z_i	z coordinate of the i^{th} path-point
$x_o, z_o, \dot{x}_o, \dot{z}_o$	adjusted values of unknowns

x_o^o, z_o^o, $\dot{x}_o^o$, z_o^o	approximate values of x_o, z_o, $\dot{x}_o$, $\dot{z}_o$
v_x, v_z, $v_{\dot{x}}$, $v_{\dot{z}}$	adjustments for x_o^o, z_o^o, $\dot{x}_o^o$, $\dot{z}_o^o$ to obtain x_o, z_o, $\dot{x}_o$, $\dot{z}_o$
Q_{ij}	observed total cycle counts.

<u>Units</u> lengths: miles (statute)
time: seconds
angles: degrees

APPENDIX B

GENERATION OF SYNTHETIC ORBIT DATA

B 3 Selection of Period of Cycle Count

Time of entry is artificially determined by incrementing time until an orbit point is found which lies inside the inner beam. (A point at which the vehicle is acquired by all receivers.) This will not, except in most unusual cases, be the actual time of entry.

B 3 1 Calculations of Δt for $\emptyset = 0\ (15)\ 75^{\circ}$

Calculate

$$(R + h) = \left[(x_o - 2a)^2 + (z_o - b')^2 \right]^{1/2}$$

$$\omega = \frac{1.24 \times 10^{-3}}{\left(\frac{R + h}{R}\right)^{3/2}}$$

$$t^o = \frac{1}{\omega} \sin^{-1} \frac{(z_o - c)\tan\psi}{(R + h)\cos\emptyset}$$

$$t^i = t^{i-1} - \Delta \quad (\text{in this case } \Delta = .2 \text{ sec})$$

Determine the smallest s such that

$$y(t^s) \leq \left[z(t^s) - c \right] (\tan\psi)$$

where

$$y(t) = -(R + h)\cos\emptyset \sin\omega t$$

$$z(t) = b' + (x_o - 2a)\sin\emptyset \sin\omega t + (-b' + z_o)\cos\omega t$$

Let

$$\Delta t = t^s/7$$

Let

$$t_i = (i - 8)\,\Delta t \qquad 1 \leq i \leq 7$$

$$= (i - 7)\,\Delta t \qquad 8 \leq i \leq 14$$

$$= 0 \qquad i = 0$$

B 3 2 Calculation of Δt for $\emptyset = 90^{\circ}$

For each of the 7 sets

$$x_o = 400 \text{ miles } \emptyset = 90^{\circ}$$

$$z_o = 150(150)1050 \text{ miles}$$

Calculate $(R + h) = \left[(x_o - 2a)^2 + (z_o - b')^2 \right]^{1/2}$

$$\omega = \frac{1.24 \times 10^{-3}}{\left(\frac{R+h}{R}\right)^{3/2}}$$

$$\Delta t = \frac{1}{\omega} \tan^{-1} \frac{2a}{z_o - b'}$$

B 4 Orbit Generation Equations

B 4 1 Position

Given "a" and assuming a constant earth radius R to each of the five stations

$$b = \left(R^2 - a^2\right)^{1/2} - \left(R^2 - (2a)^2\right)^{1/2}$$

$$b' = -\left(R^2 - (2a)^2\right)^{1/2}$$

$$c = \left(R^2 - a^2\right)^{1/2} - \left(R^2 - (2a)^2\right)^{1/2}$$

See Figure 5

Compute 45 values i = 0, ... 14

$$x_i = 2a + (-b' + z_o) \sin \emptyset \sin \omega t_i + (x_o - 2a) \cos \omega t_i$$

$$y_i = (R + h) \cos \emptyset \sin \omega t_i$$

$$z_i = b' + (x_o - 2a) \sin \emptyset \sin \omega t_i + (-b' + z_o) \cos \omega t_i$$

B 4 2 Velocity

Compute the two values

$$\dot{x}_o = \omega (-b' + z_o) \sin \emptyset$$

$$\dot{z}_o = \omega (x_o - 2a) \sin \emptyset$$

where $\dot{x}_o = \dot{x}(x_o, z_o)$

B 5 Generation of Range Sums

Let ρ_{ij} denote range-sum. Compute the 56 values

$0 \le i \le 14 \qquad 1 \le j \le 4$

$$\rho_{ij} = \left[x_i^2 + y_i^2 + z_i^2\right]^{1/2} + \left[(x_i - X_j)^2 + y_i^2 + (z_i - Z_j)^2\right]^{1/2}$$

Compute the 5 values

$$r_o = \left[x_o^2 + z_o^2\right]^{1/2} \qquad (x_o, 0, z_o) \text{ is a grid point}$$

$$r_j = \left[(x_o - X_j)^2 + (z_o - Z_j)^2\right]^{1/2} \qquad 1 \le j \le 4$$

B.6 Range Sum Difference

Let

$$\Delta_{ij} = \rho_{ij} - \rho_{oj}$$

B.6.1 Introduction of Error into "True" Range-Sum Differences

In order to simulate realistic conditions, errors in Δ_{ij} may be introduced. This is usually done either by means of fixed bias errors or random error generation.

In any case let

$$M_{ij} = \Delta_{ij} + \text{error}_{ij}$$

In the error less case, of course,

$$\Delta_{ij} = M_{ij}$$

APPENDIX C

ILLUSTRATIONS

APPENDIX C

LIST OF ILLUSTRATIONS

Figure Number	Title
1	Flow Chart: Synthetic Data Generation
2a	Flow Chart. Data Reduction - Position
2b	Flow Chart: Data Reduction - Velocity
3	Flow Chart: Test Case
4	Beam Configuration
5	Station Configuration
6abcd to 14abcd	A plot of $\frac{\sigma_x}{\sigma_M}, \frac{\sigma_z}{\sigma_M}, \frac{\sigma_x}{\sigma_M}, \frac{\sigma_z}{\sigma_M}$ vs Beamwidth for fixed x and Ø, for constant contours of z.
15ab to 20ab	A plot of $\frac{\sigma_x}{\sigma_M}, \frac{\sigma_z}{\sigma_M}$, vs z for fixed x and Beamwidth, for constant contours of Ø
21ab to 28ab	A plot of $\frac{\sigma_x}{\sigma_M}, \frac{\sigma_z}{\sigma_M}$, vs x for fixed Beamwidth and Ø, for constant contours of z
29ab to 34 ab	A plot of $\frac{\sigma_x}{\sigma_M}, \frac{\sigma_z}{\sigma_M}, \frac{\sigma_x}{\sigma_M}, \frac{\sigma_{\dot{z}}}{\sigma_M}$ vs z for fixed x and Ø, for constant contours of Beamwidth.
35abcd	A plot of $\frac{\sigma_x}{\sigma_M}, \frac{\sigma_z}{\sigma_M}, \frac{\sigma_x}{\sigma_M}, \frac{\sigma_{\dot{z}}}{\sigma_M}$ vs z for fixed x and Ø, for constant contours independent of Beamwidth.

FLOW CHART: GENERATION OF SYNTHETIC ORBIT DATA

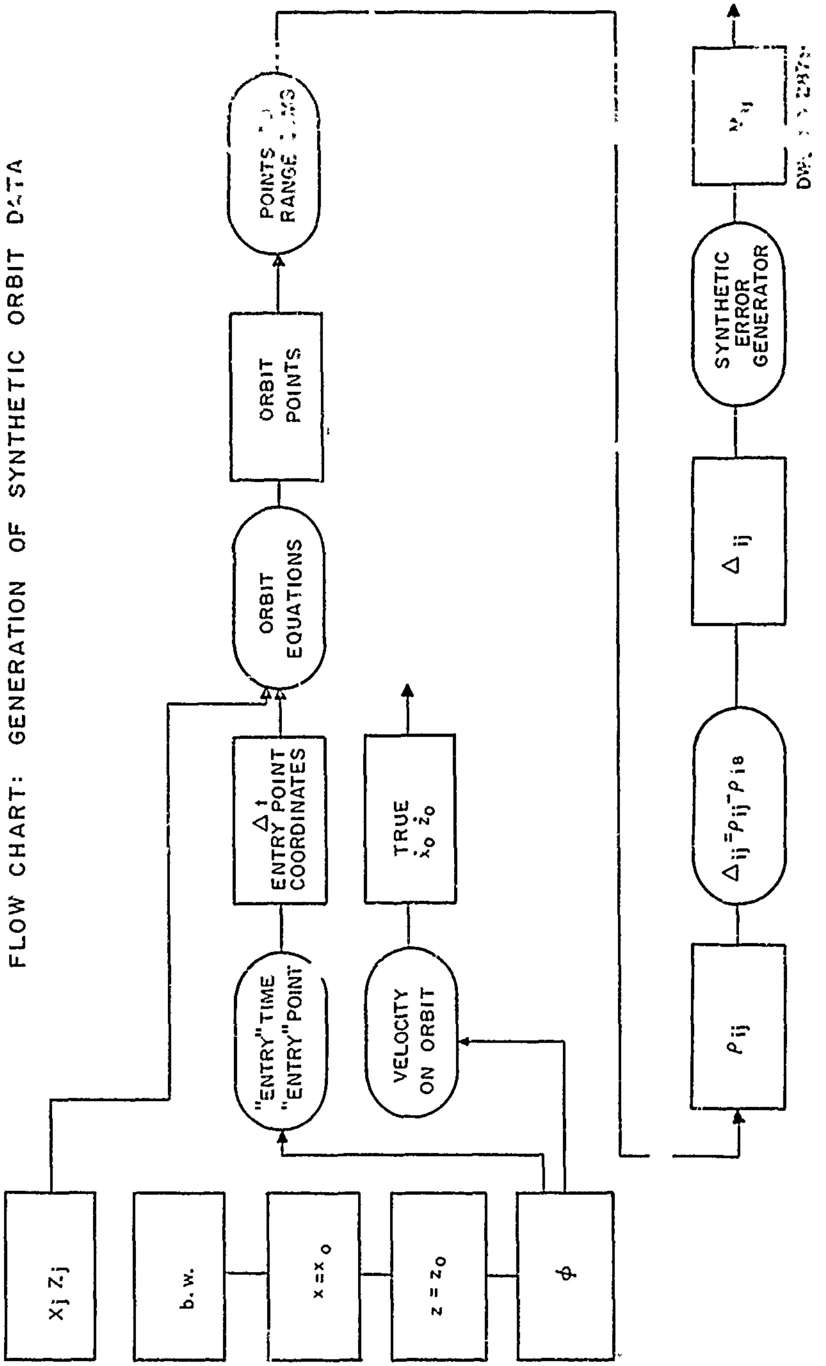
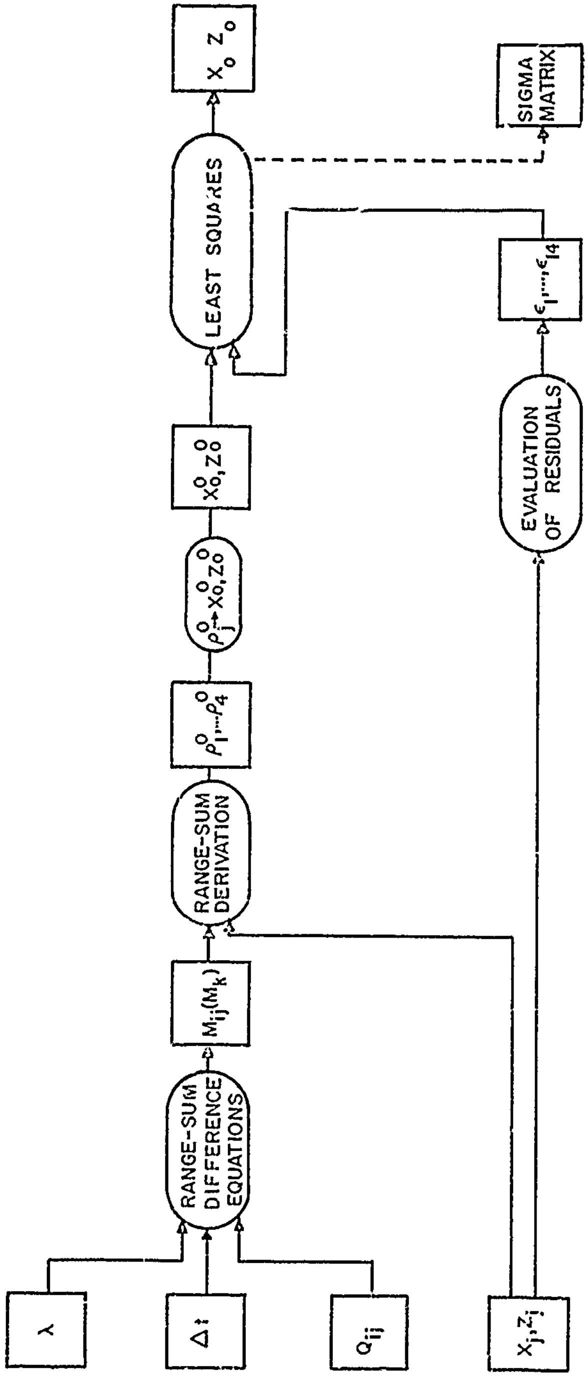


FIGURE 1

FLOW CHART: POSITION DETERMINATION



DWG NO 2881

FIGURE 2a

FLOW CHART: VELOCITY DETERMINATION

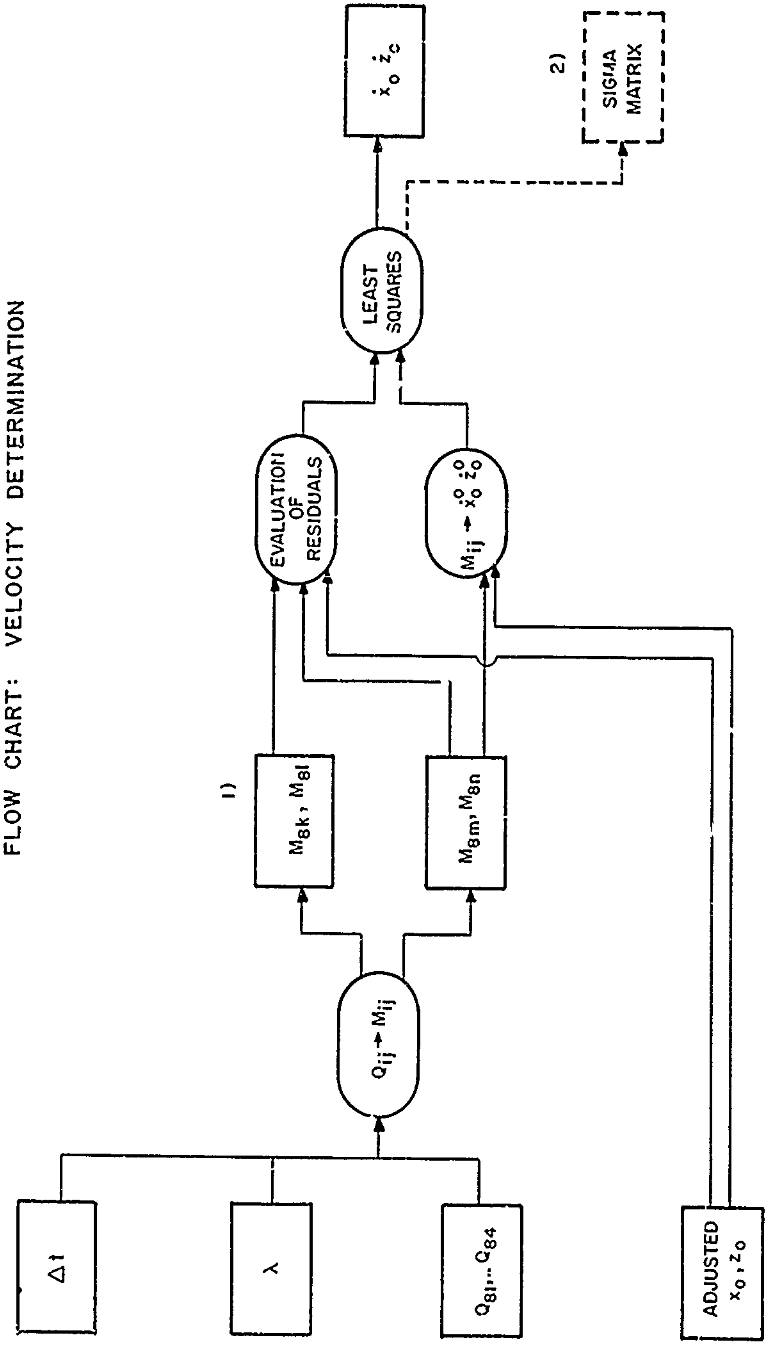


FIGURE 2b

DWG NO 2880

FLOW CHART: TEST CASE

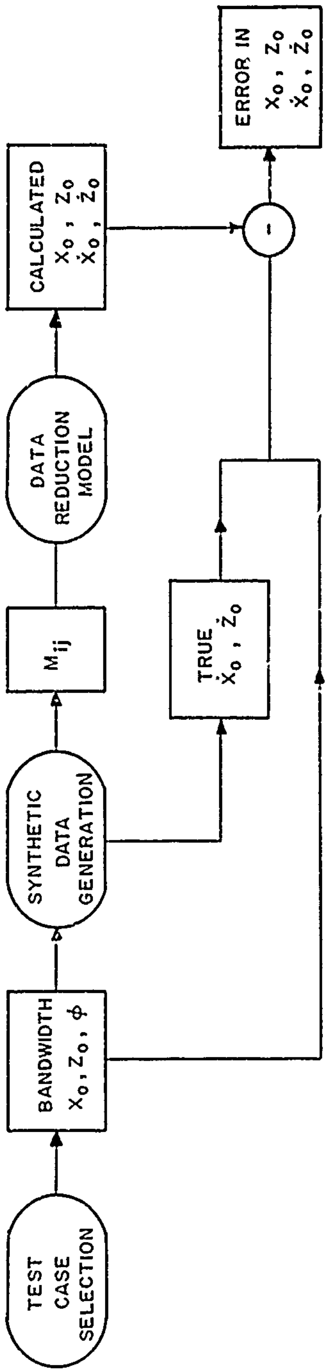
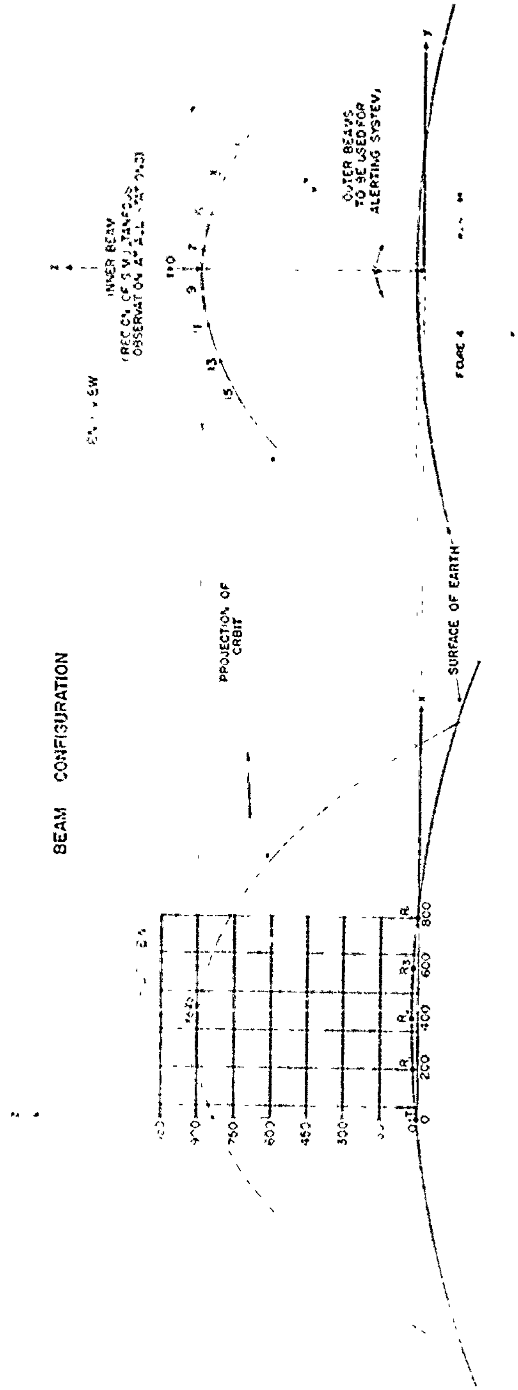


FIGURE 3

DWG NO 2878


BEAM CONFIGURATION
PROJECTION OF ORBIT
SURFACE OF EARTH
INNER BEAM
OUTER BEAMS TO BE USED FOR ALERTING SYSTEM
0
200
400
600
800
150
300
450
600
750
900

STATION CONFIGURATION

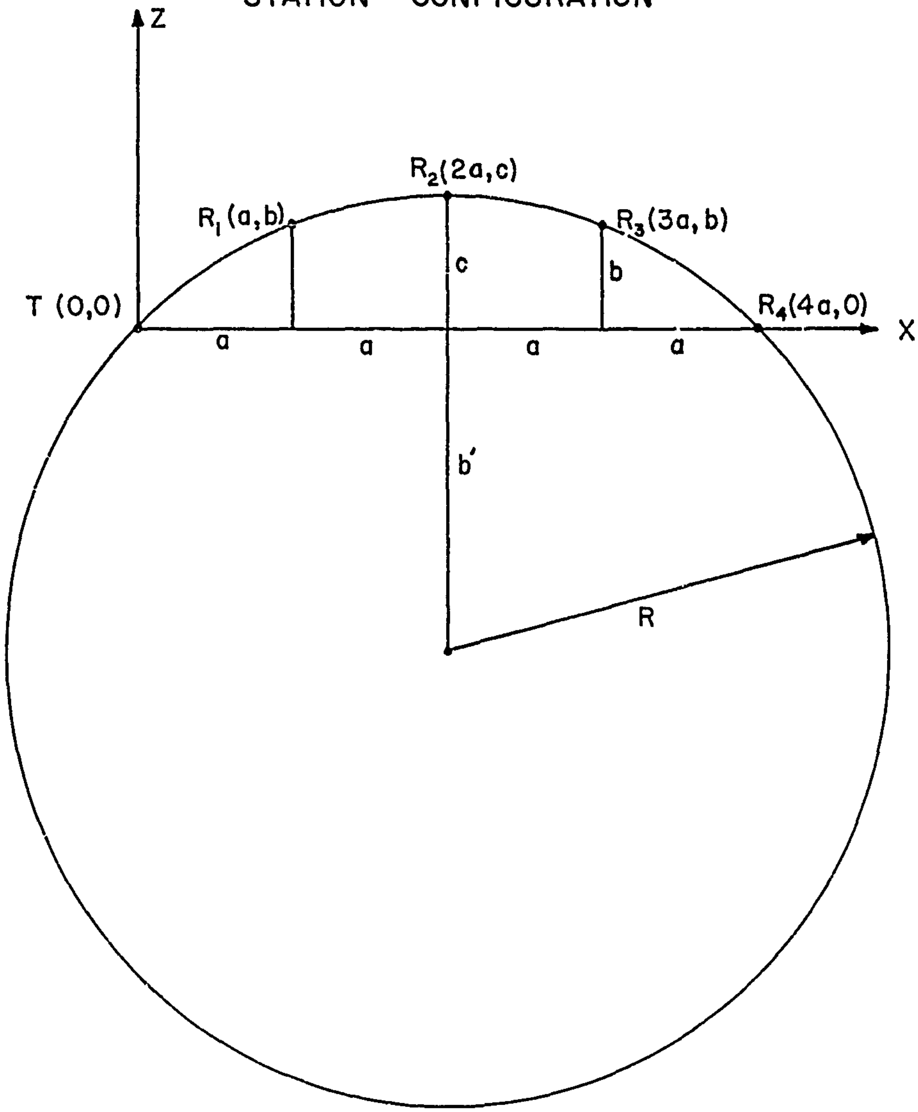


FIGURE 5

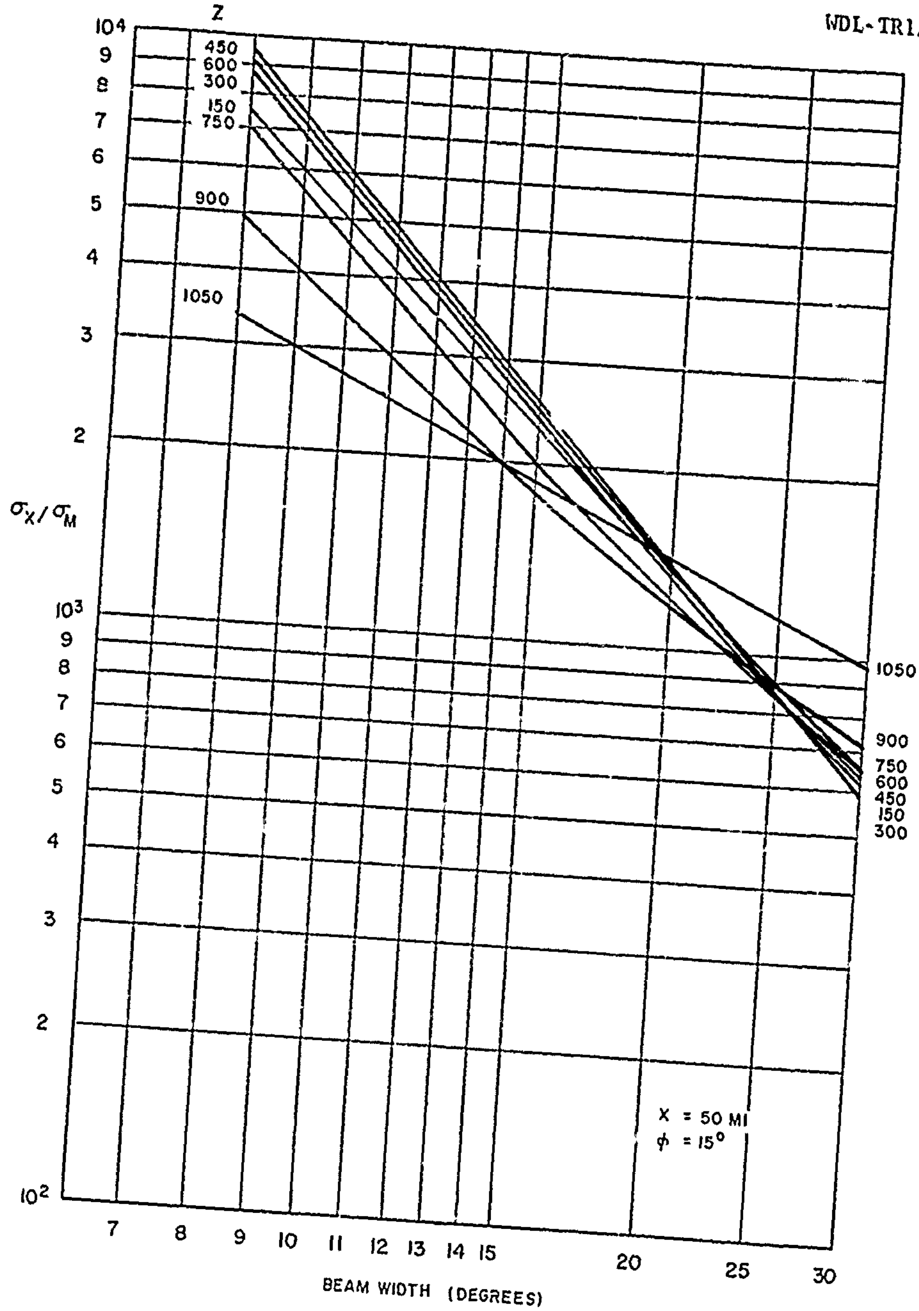


Figure 6a

DWG NO. 2604

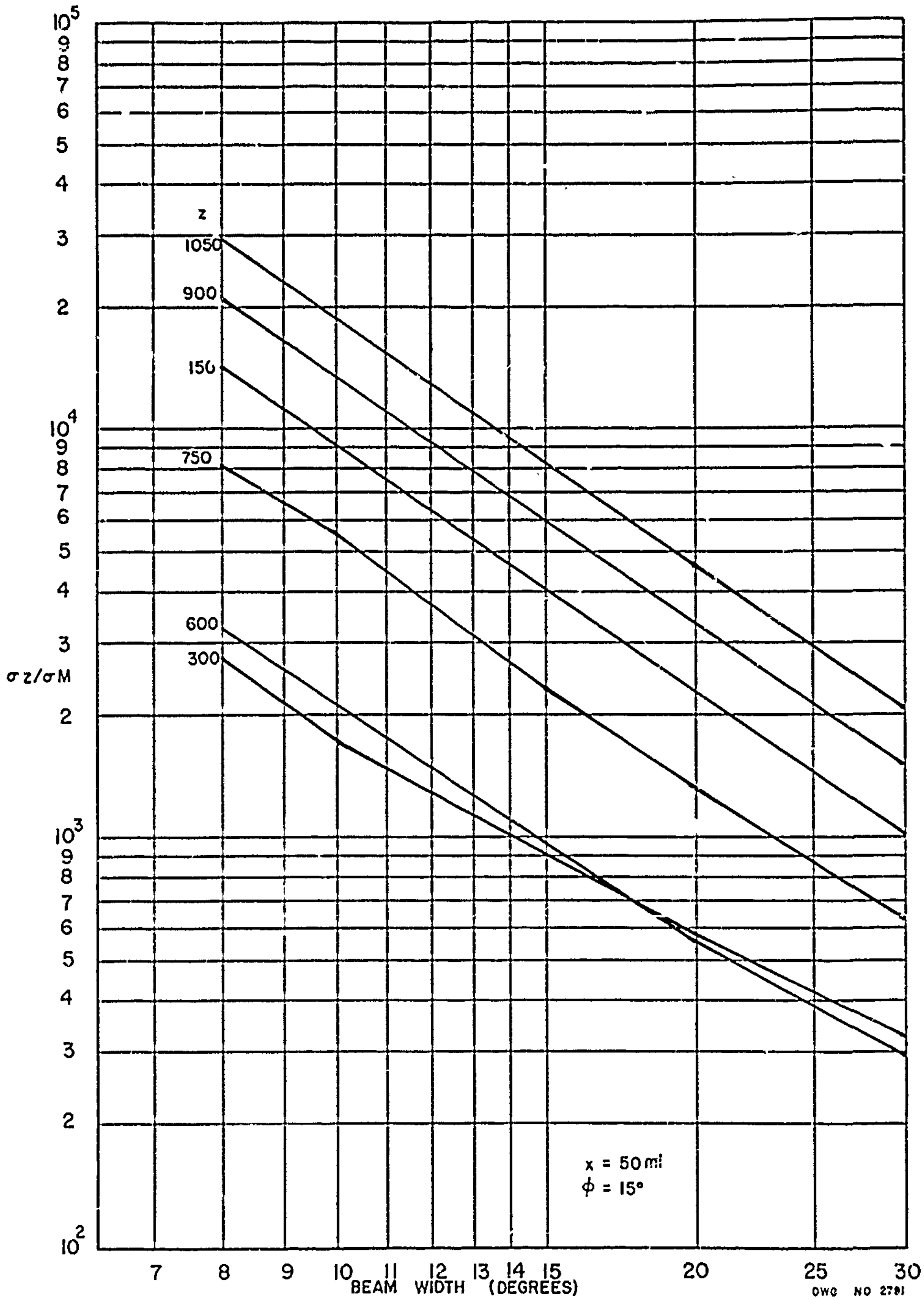

z
1050
900
150
750
600
300
σz/σM
x = 50 mi
φ = 15°
BEAM WIDTH (DEGREES)
DWG NO 2781

Figure 6b
C 9

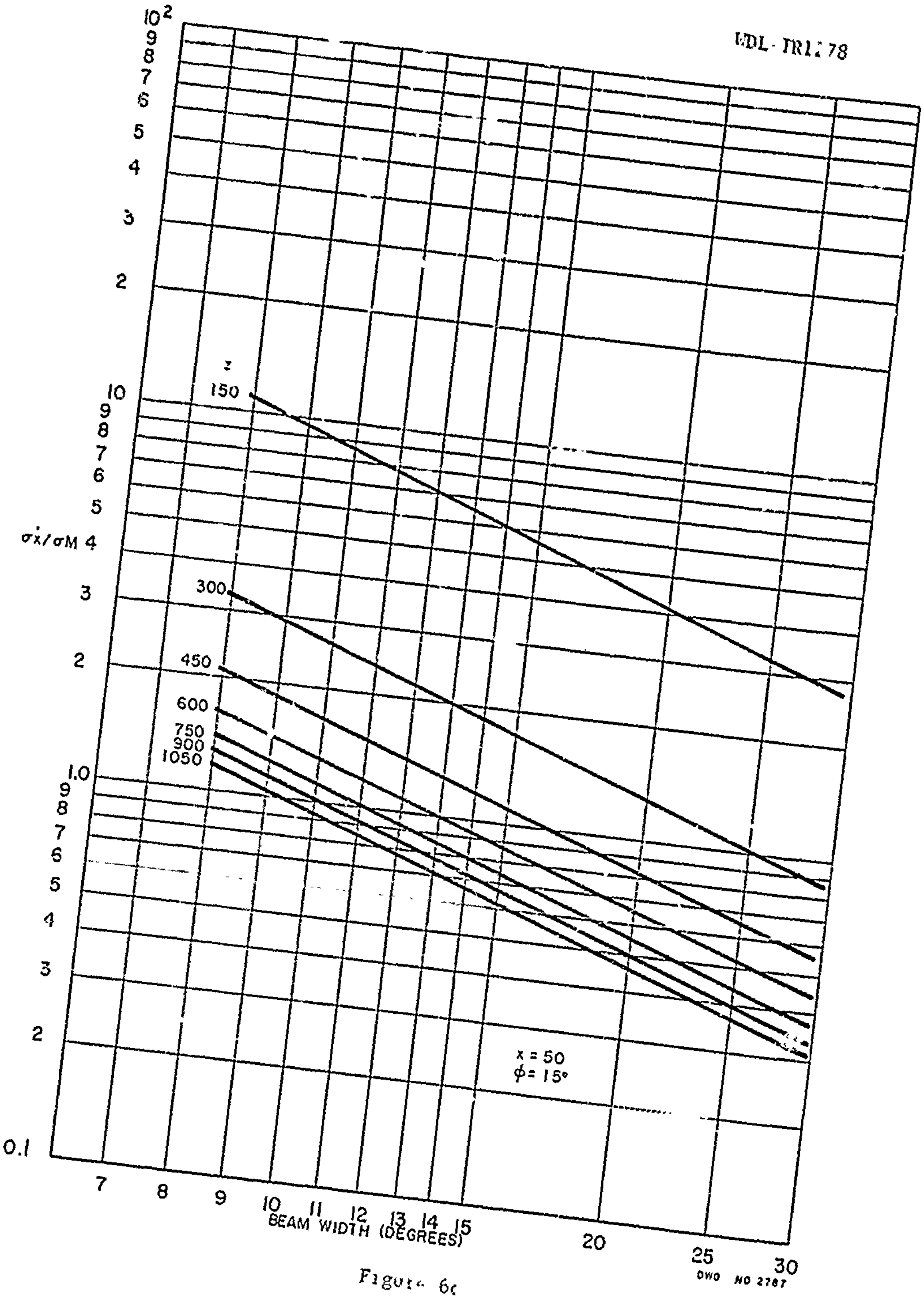
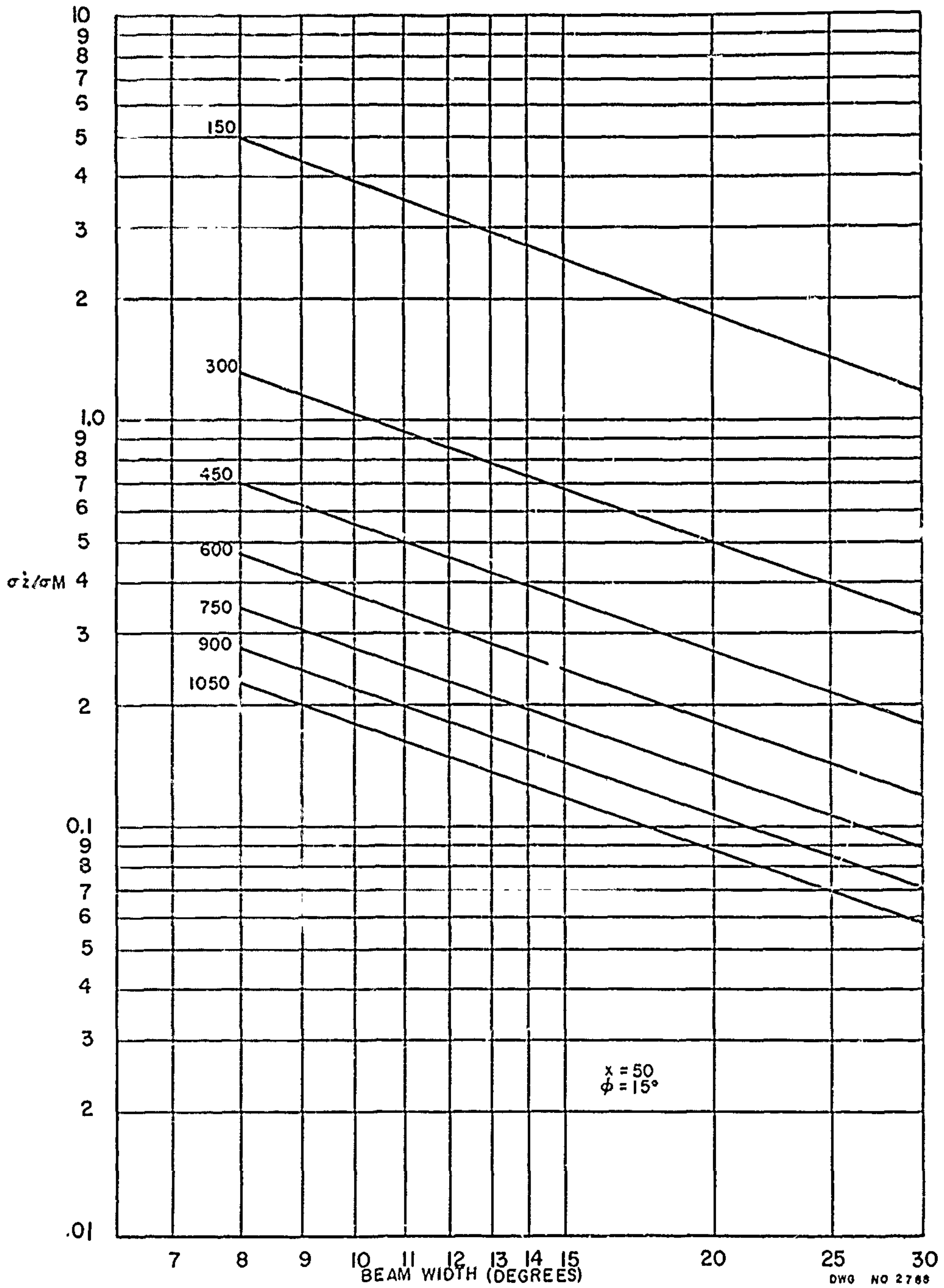

10^2
9
8
7
6
5
4
3
2
10
9
8
7
6
5
$\sigma_{\dot{x}}/\sigma_M$ 4
3
2
1.0
9
8
7
6
5
4
3
2
0.1
z
150
300
450
600
750
900
1050
x = 50
φ = 15°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2787

Figure 6c



150
300
450
600
750
900
1050
$\sigma\dot{z}/\sigma M$
x = 50
ϕ = 15°
BEAM WIDTH (DEGREES)
DWG NO 2788

Figure 6d

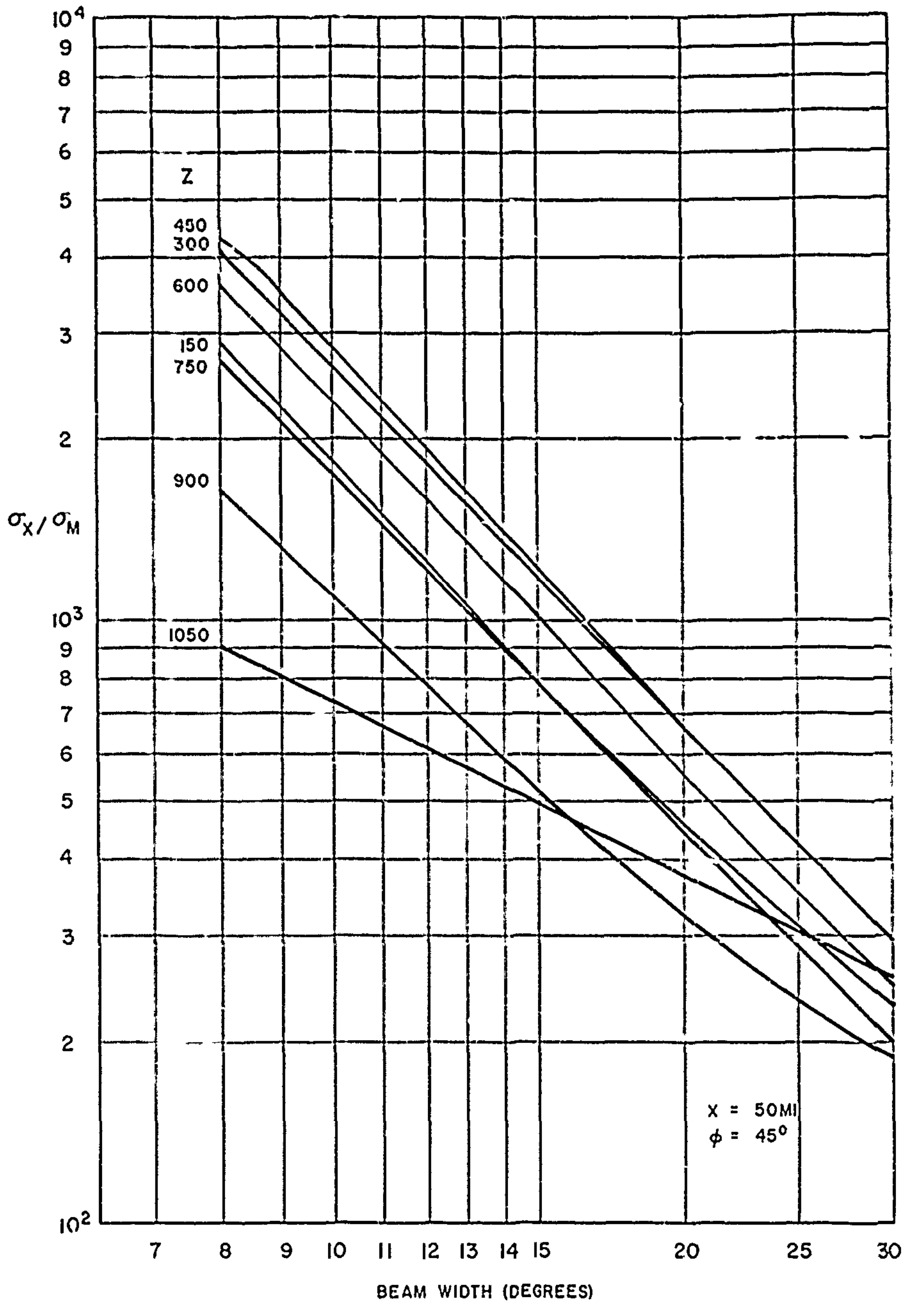


Figure 7a

DWG NO 2808

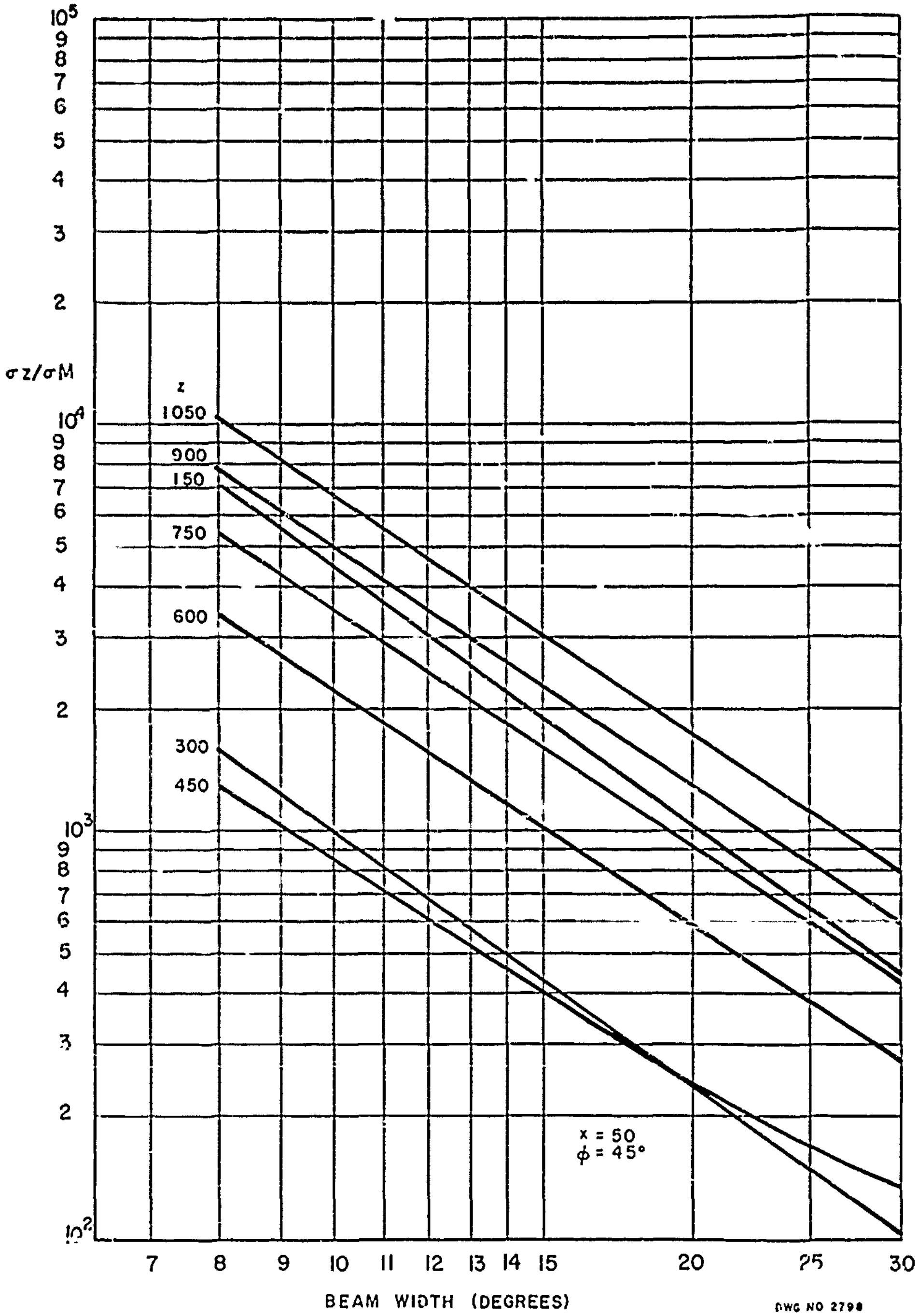

10^5
9
8
7
6
5
4
3
2
$\sigma z/\sigma M$
z
1050
900
150
750
600
300
450
10^4
10^3
10^2
x = 50
ϕ = 45°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2798

Figure 7b

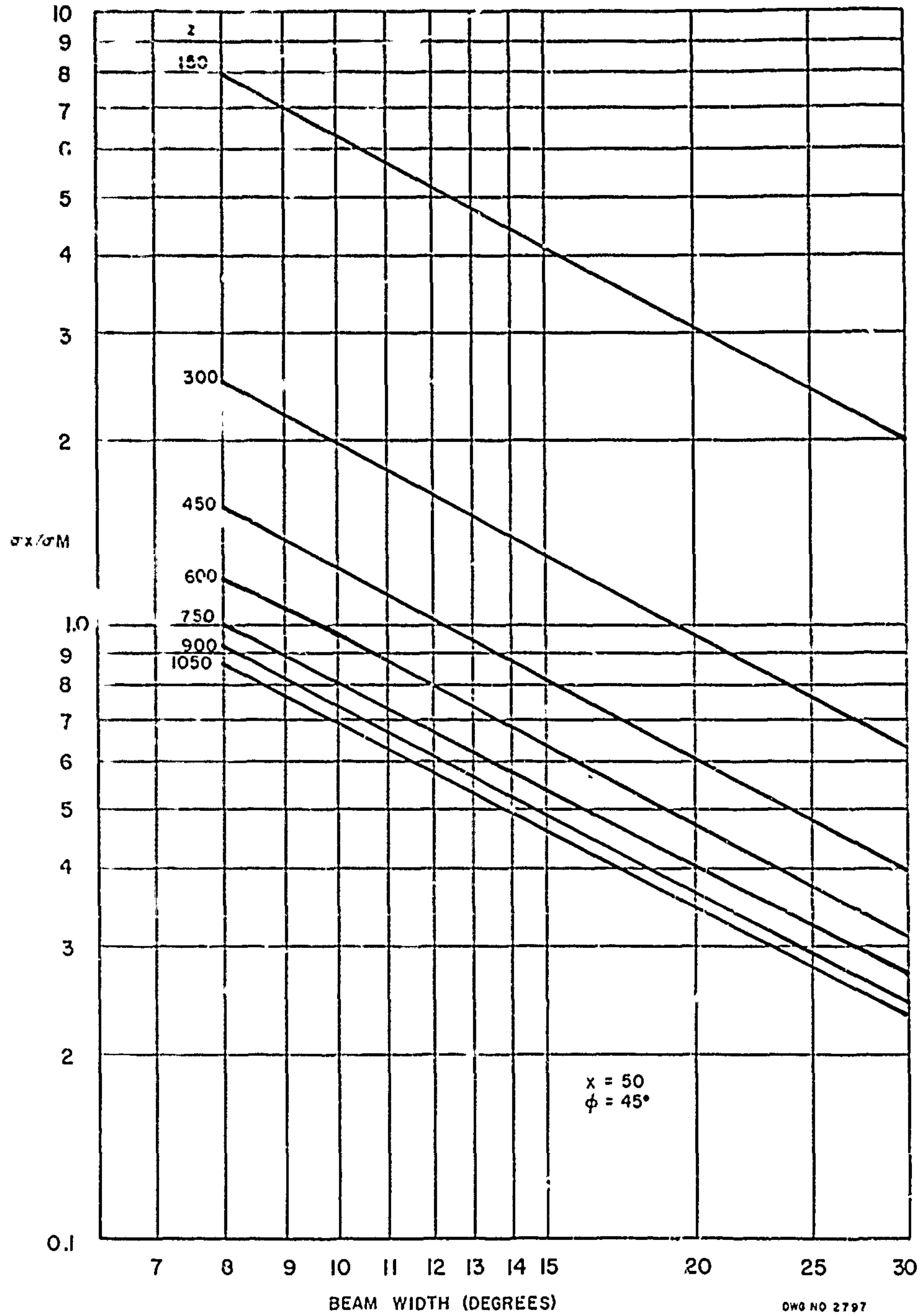

z
150
300
450
600
750
900
1050
σx/σM
10
9
8
7
6
5
4
3
2
1.0
9
8
7
6
5
4
3
2
0.1
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
x = 50
φ = 45°
DWG NO 2797

Figure 7c

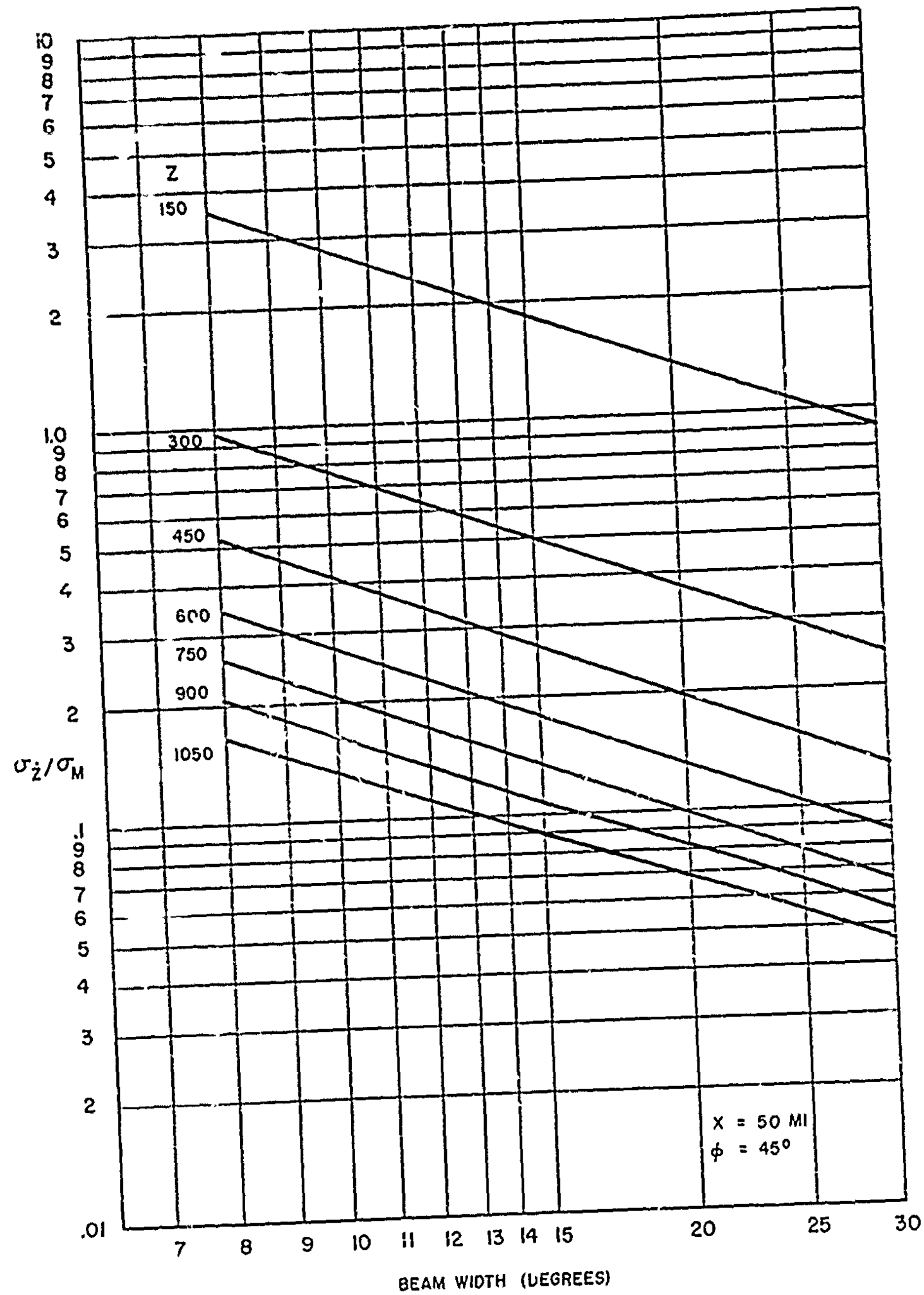


Figure 7d

DWG NO 2807

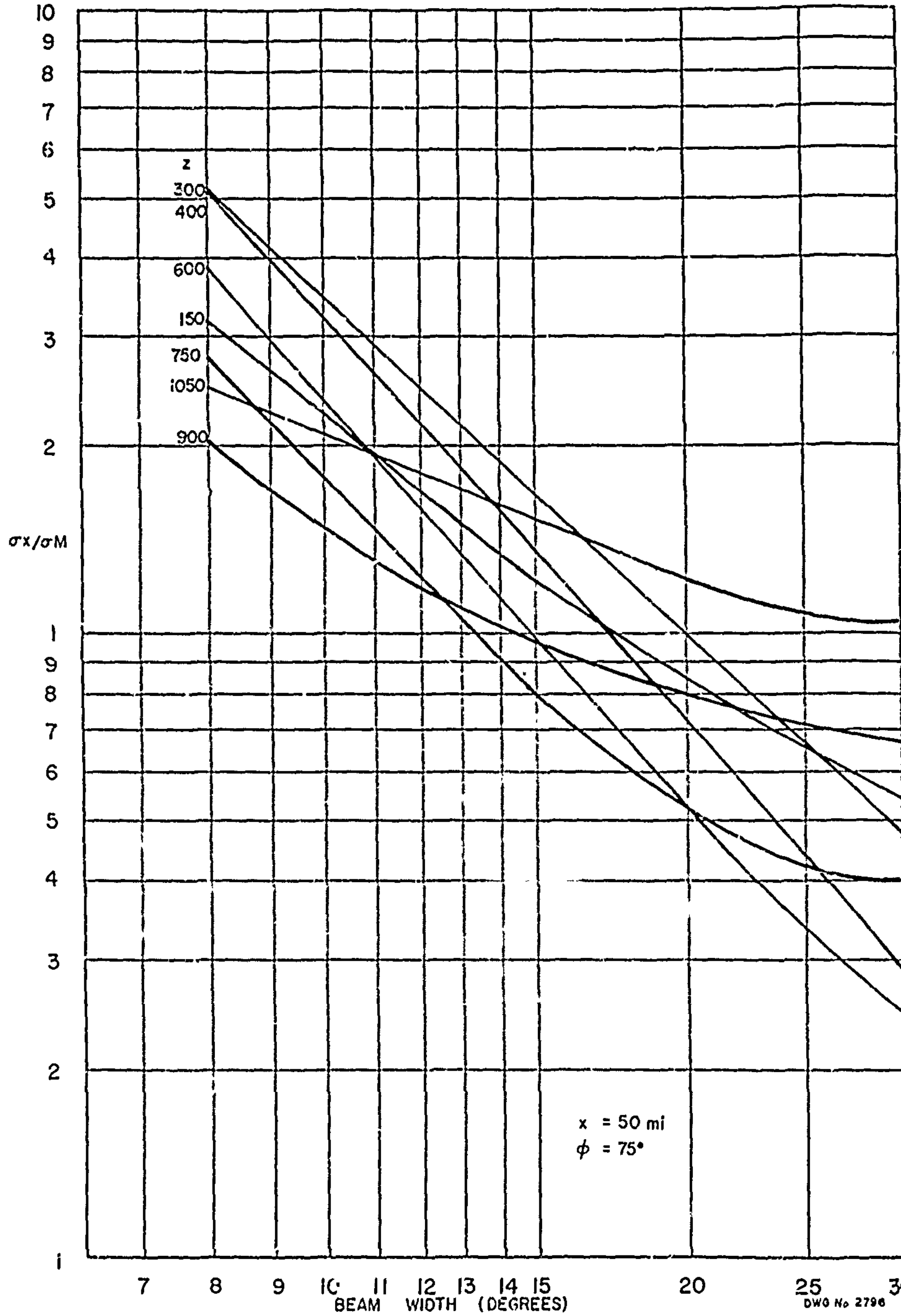

10
9
8
7
6
5
4
3
2
1
9
8
7
6
5
4
3
2
1
σx/σM
z
300
400
600
150
750
1050
900
x = 50 mi
φ = 75°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG No 2796

Figure 8a

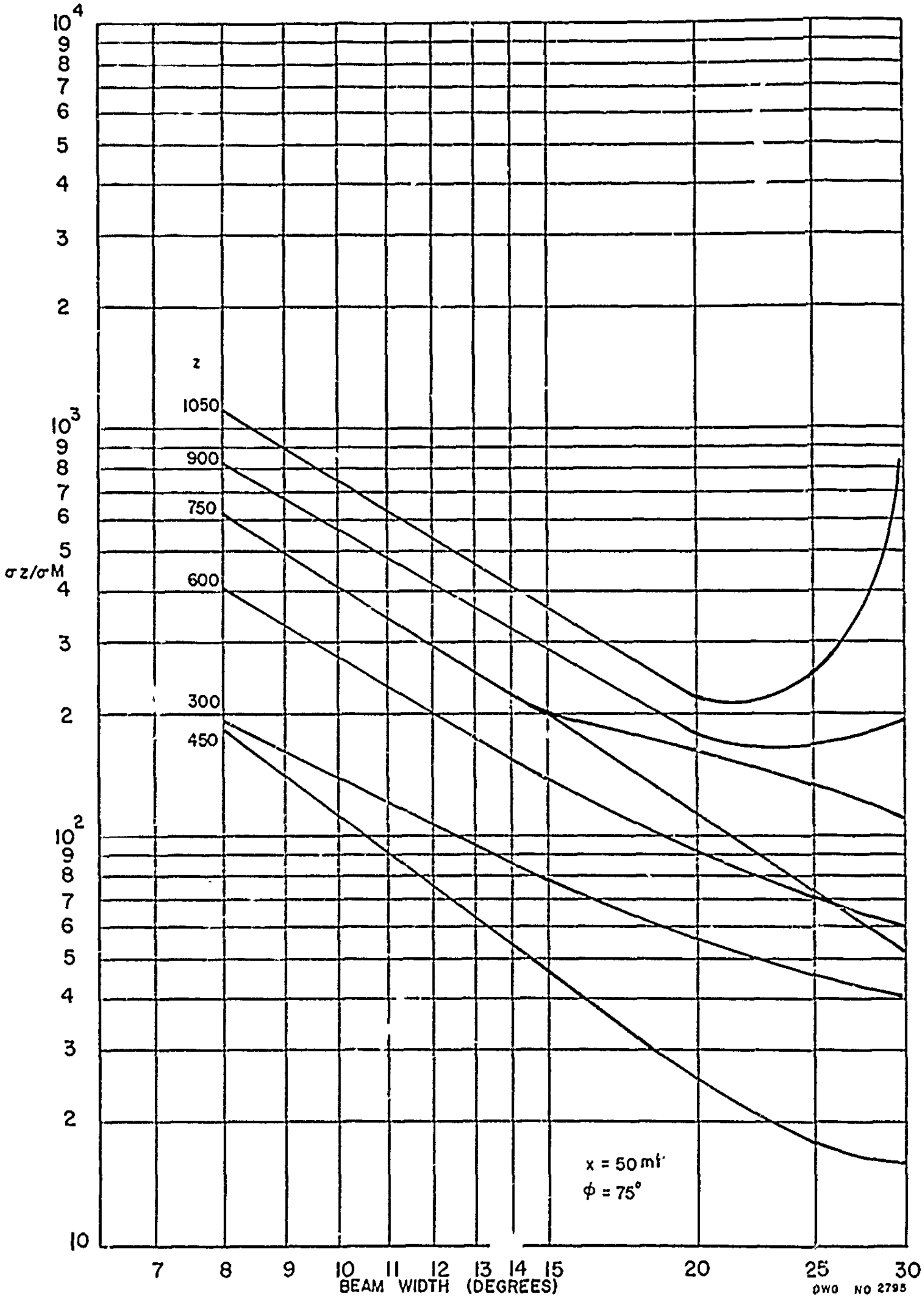

z
1050
900
750
600
300
450
σz/σM
BEAM WIDTH (DEGREES)
x = 50 mi
φ = 75°
DWG NO 2795

Figure 8b

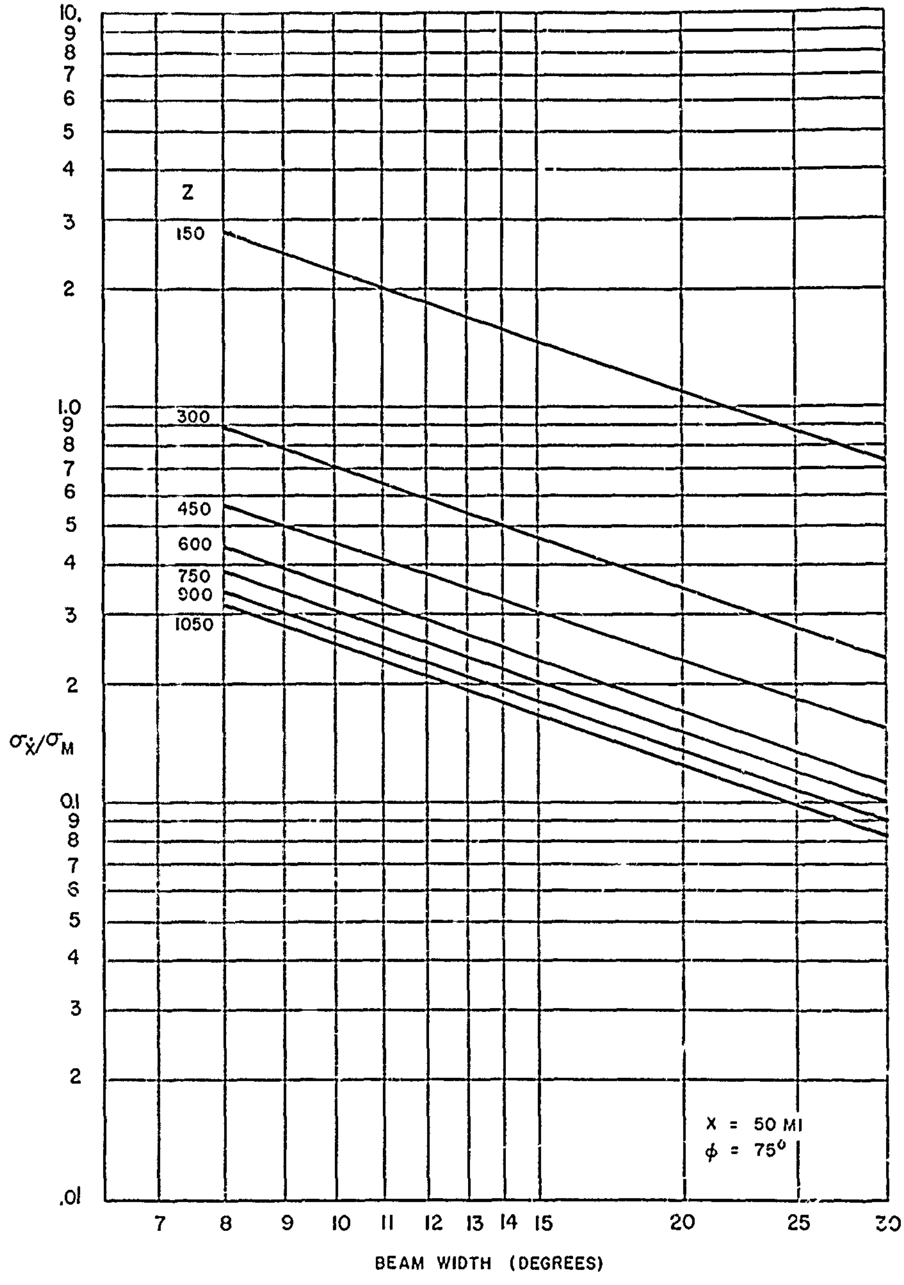


Figure 8c

DWG NO 2606

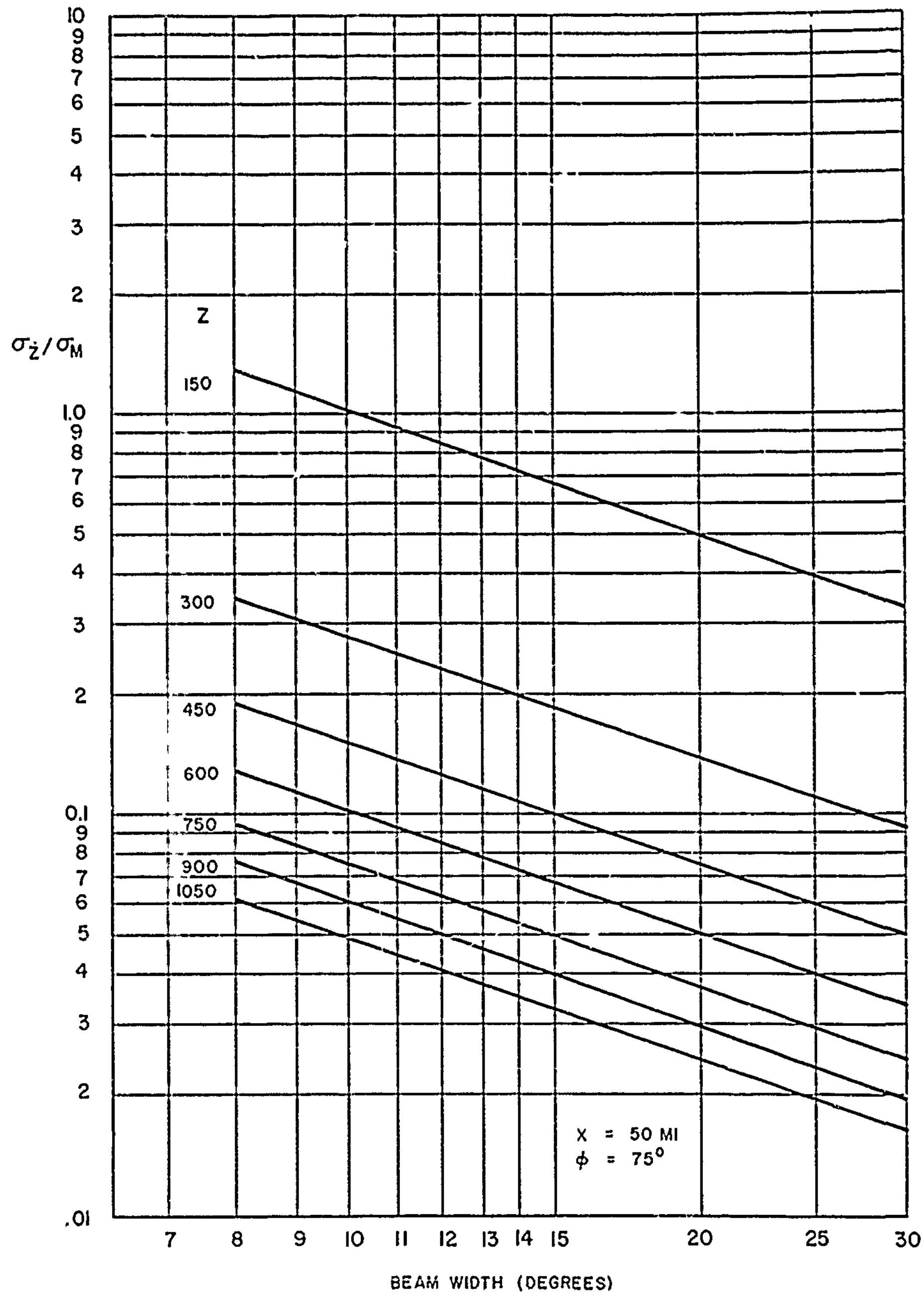
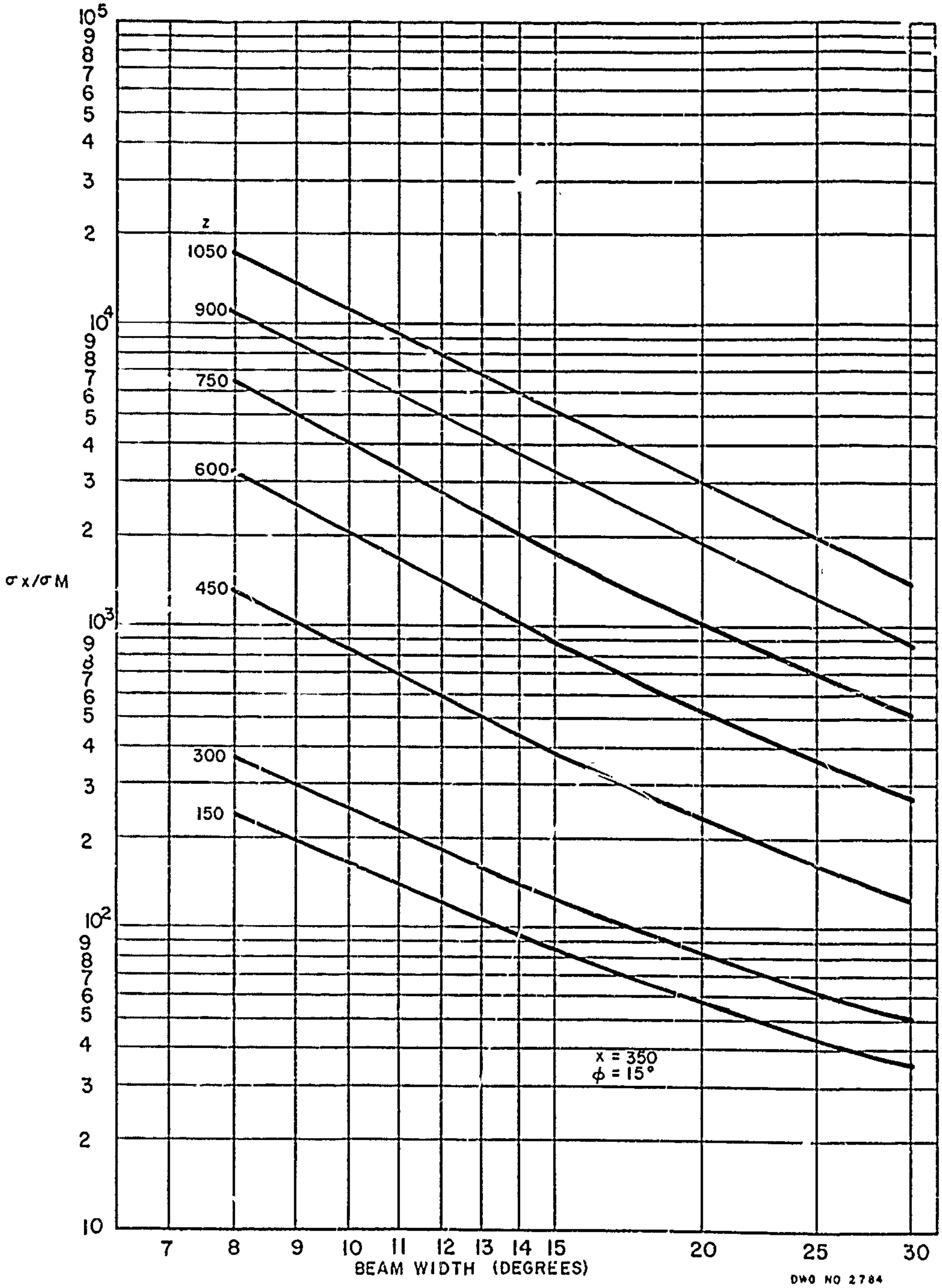


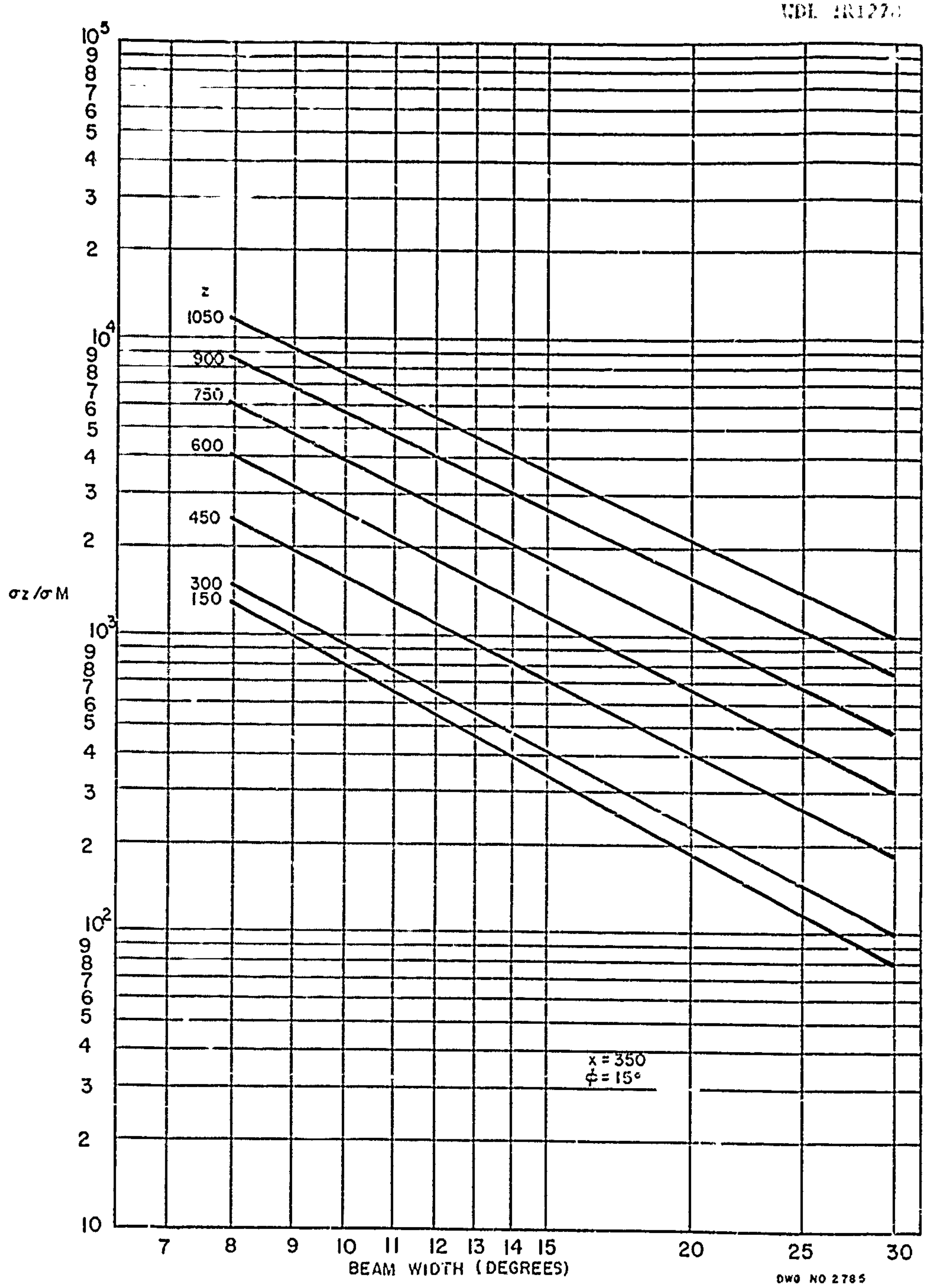
Figure 8d

DWG NO. 2806



10^5
9
8
7
6
5
4
3
2
10^4
10^3
10^2
10
$\sigma x/\sigma M$
z
1050
900
750
600
450
300
150
x = 350
ϕ = 15°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2784

Figure 9a



σz/σM
10^5
10^4
10^3
10^2
10
z
1050
900
750
600
450
300
150
x = 350
Φ = 15°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2785

Figure 9b

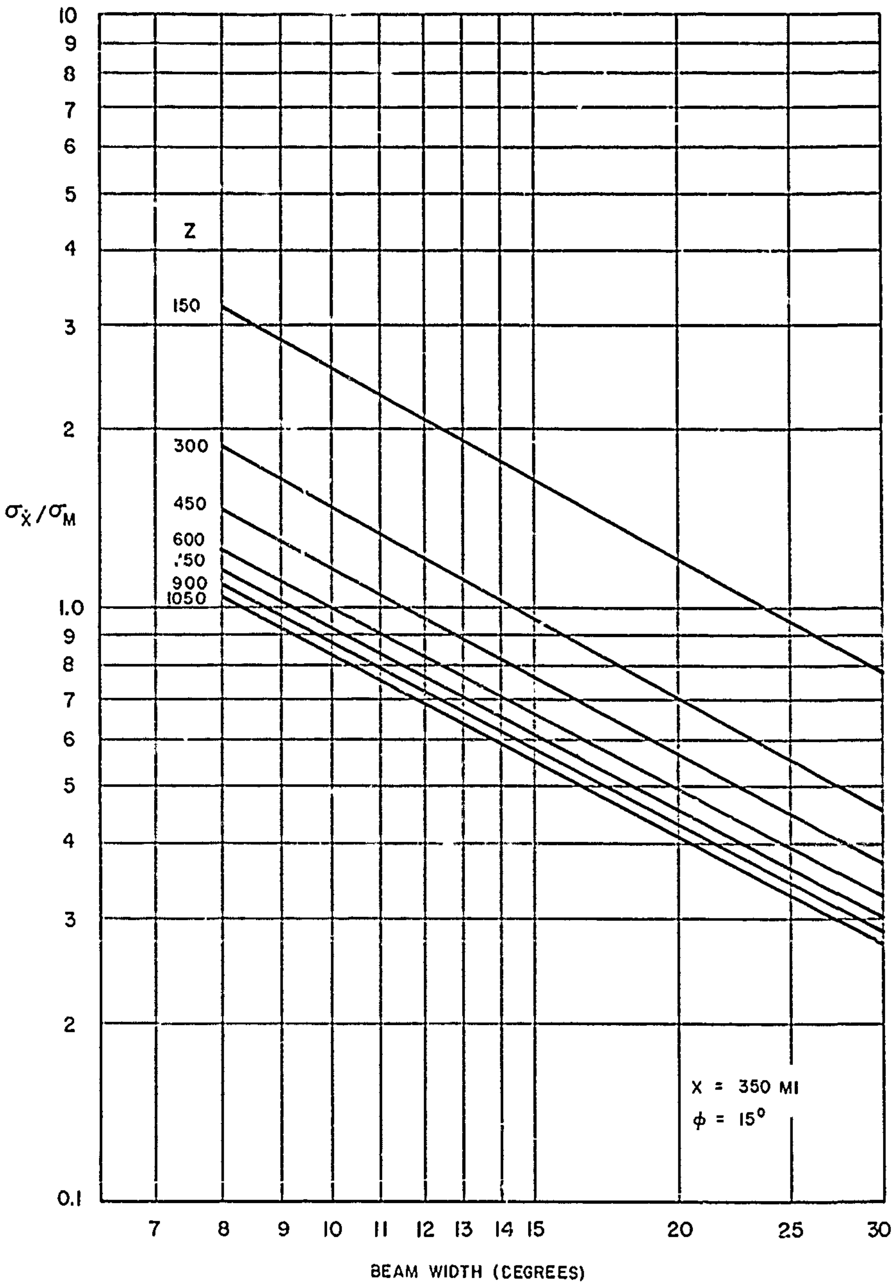
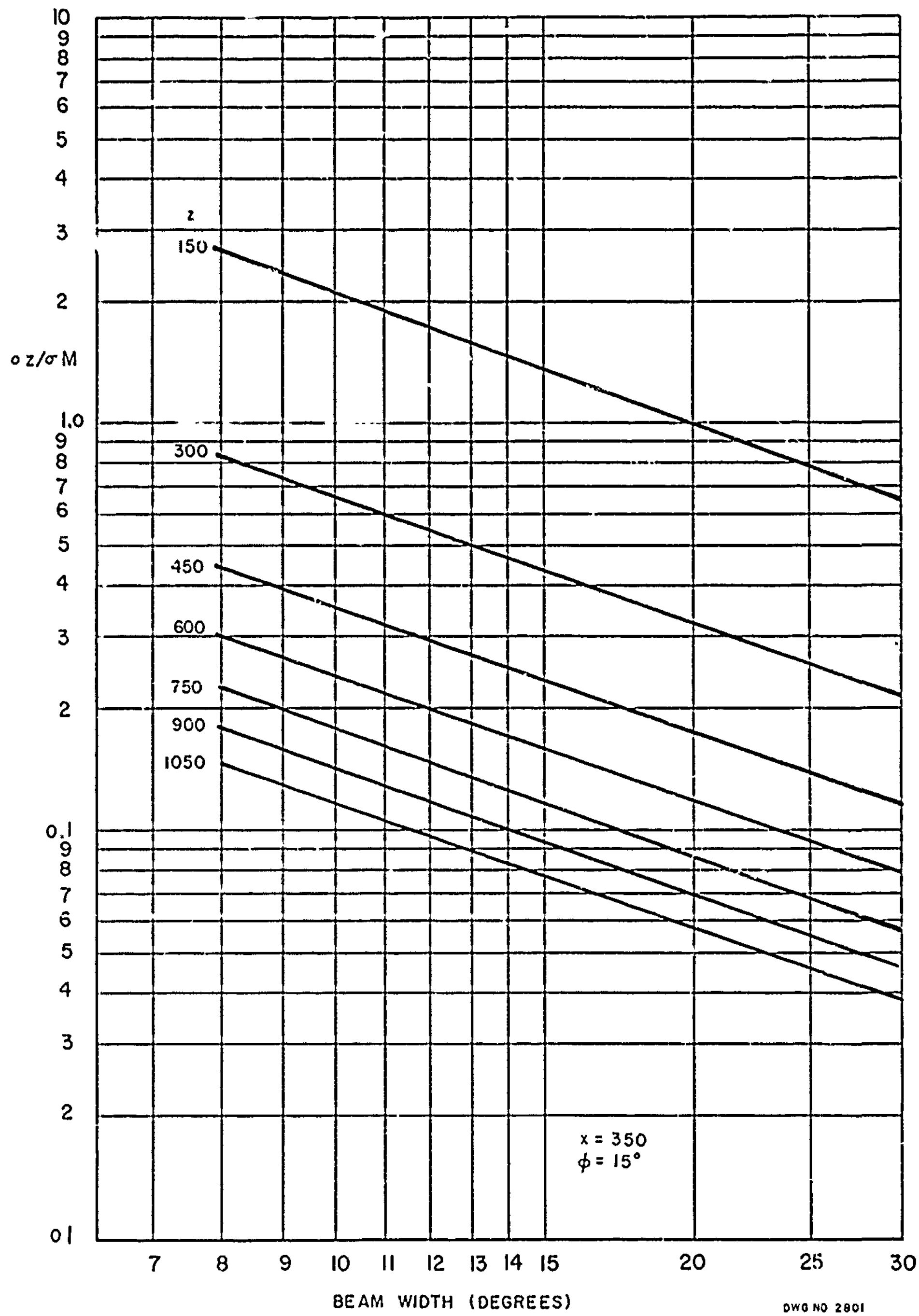


Figure 9c

DWG NO. 2810



10
9
8
7
6
5
4
3
2
1.0
0.1
01
σz/σM
z
150
300
450
600
750
900
1050
x = 350
φ = 15°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2801

Figure 9d

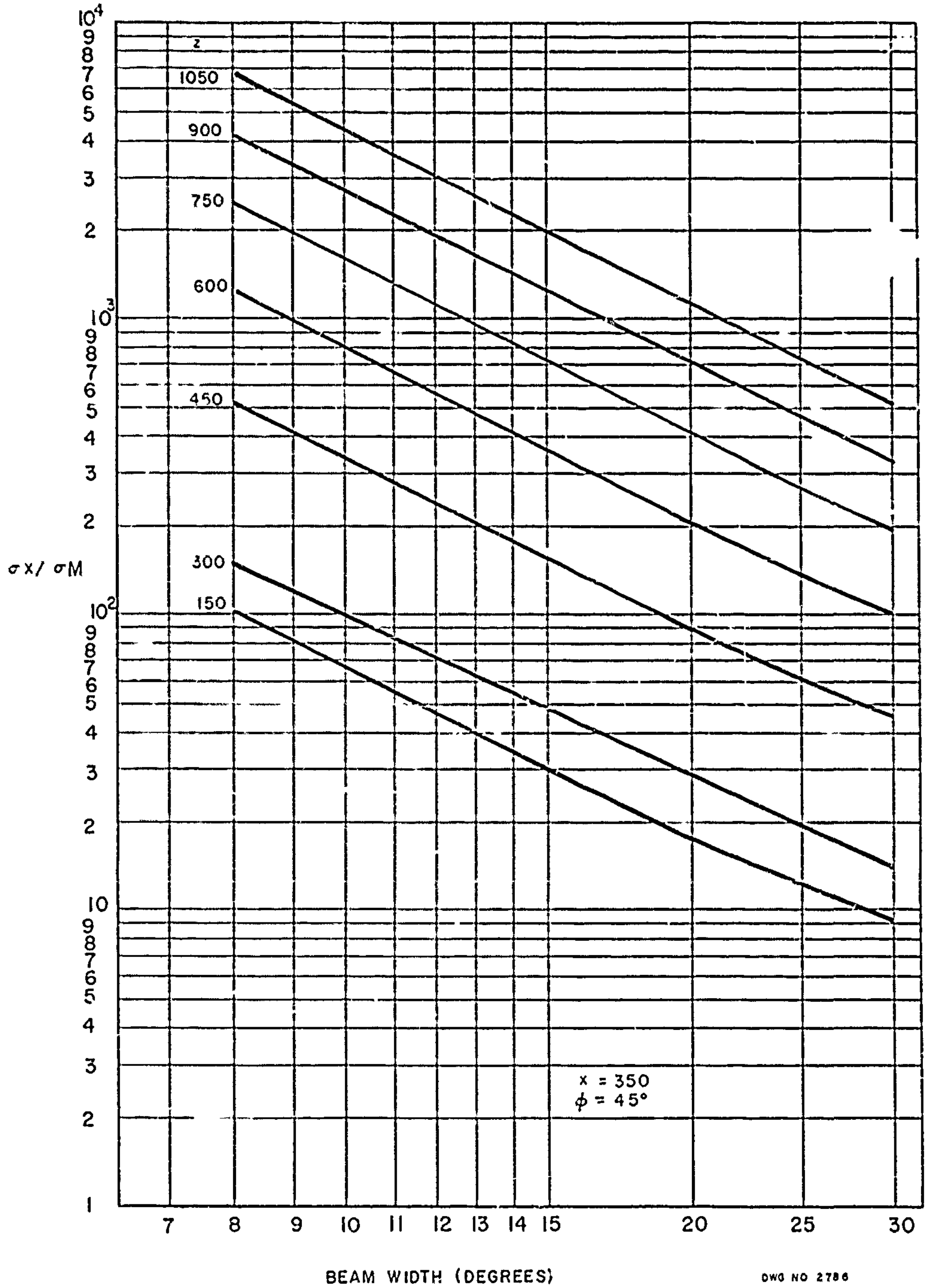
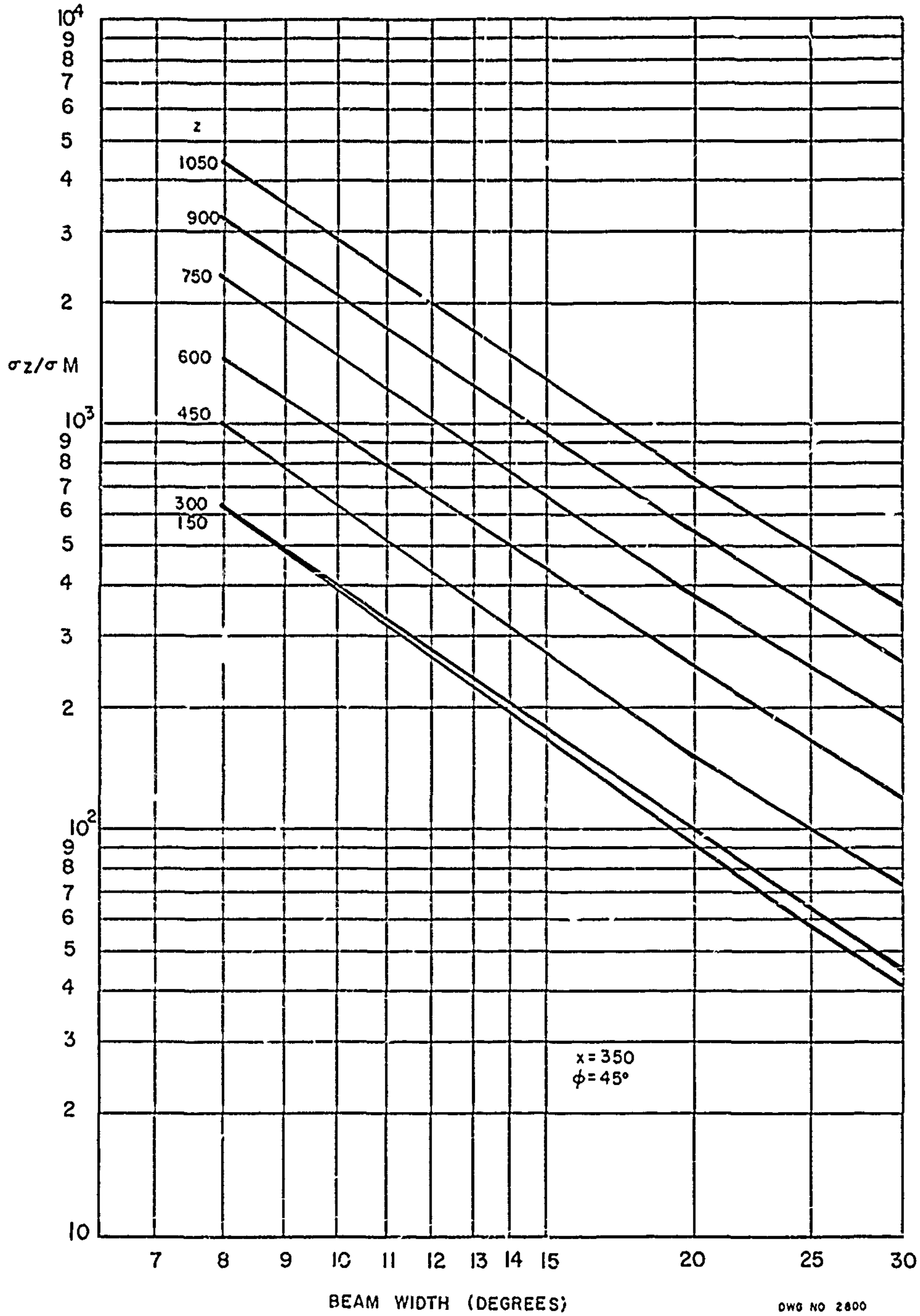

10^4
9
8
7
6
5
4
3
2
10^3
9
8
7
6
5
4
3
2
10^2
9
8
7
6
5
4
3
2
10
9
8
7
6
5
4
3
2
1
σx/σM
z
1050
900
750
600
450
300
150
x = 350
φ = 45°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2786

Figure 10a



10^4
9
8
7
6
5
4
3
2
$\sigma z/\sigma M$
10^3
10^2
10
z
1050
900
750
600
450
300
150
x = 350
$\phi = 45°$
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2800

Figure 10b

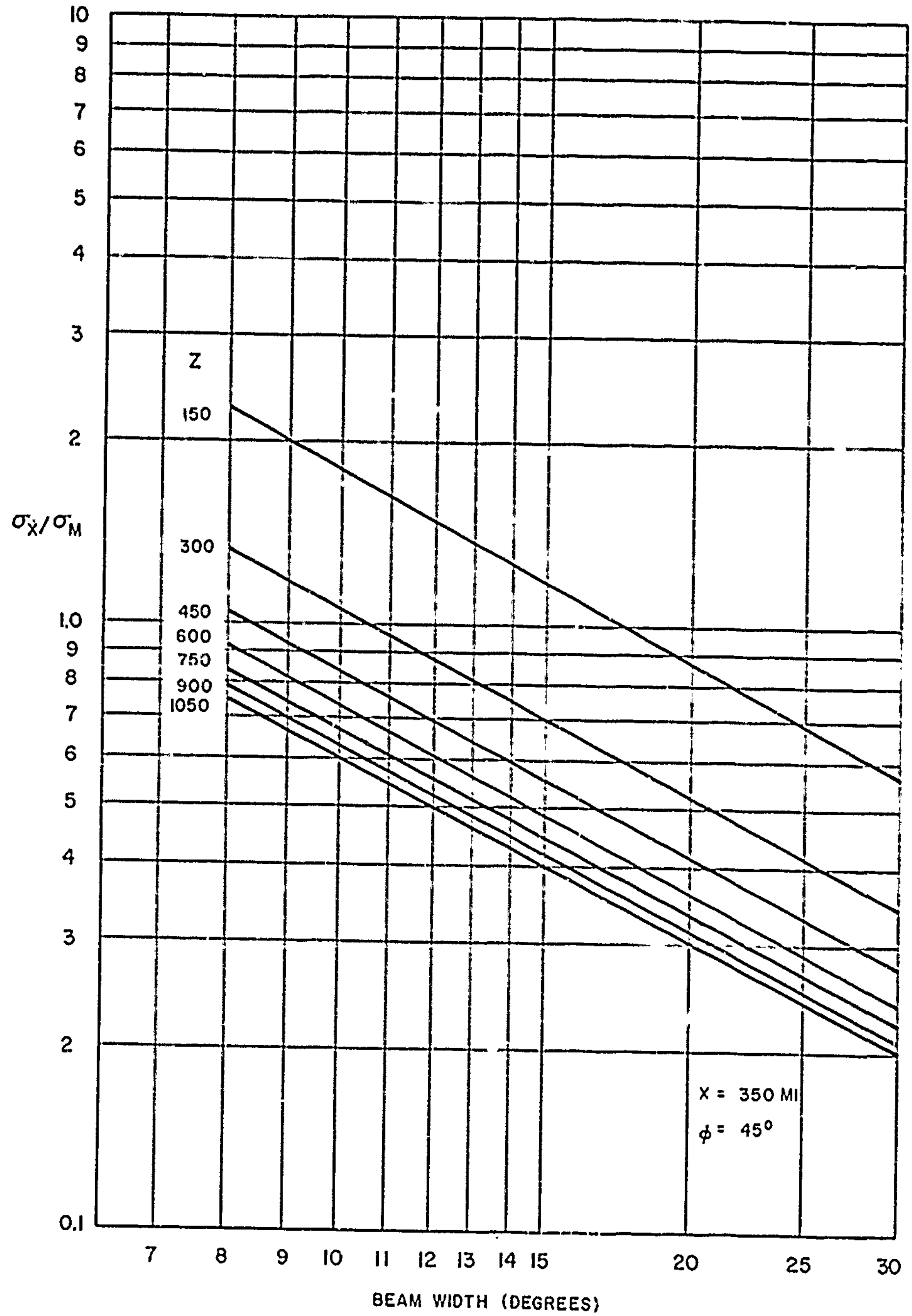
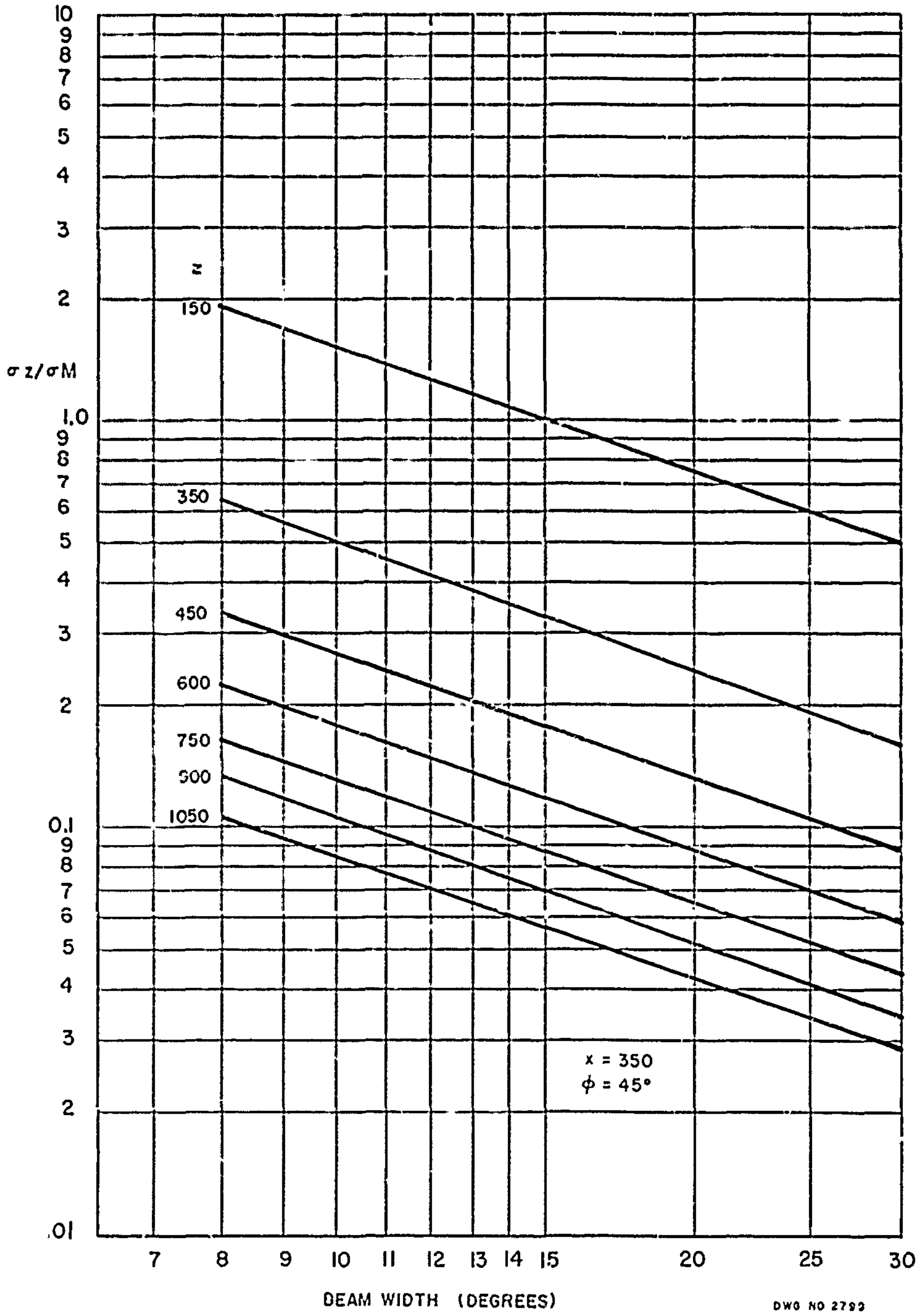


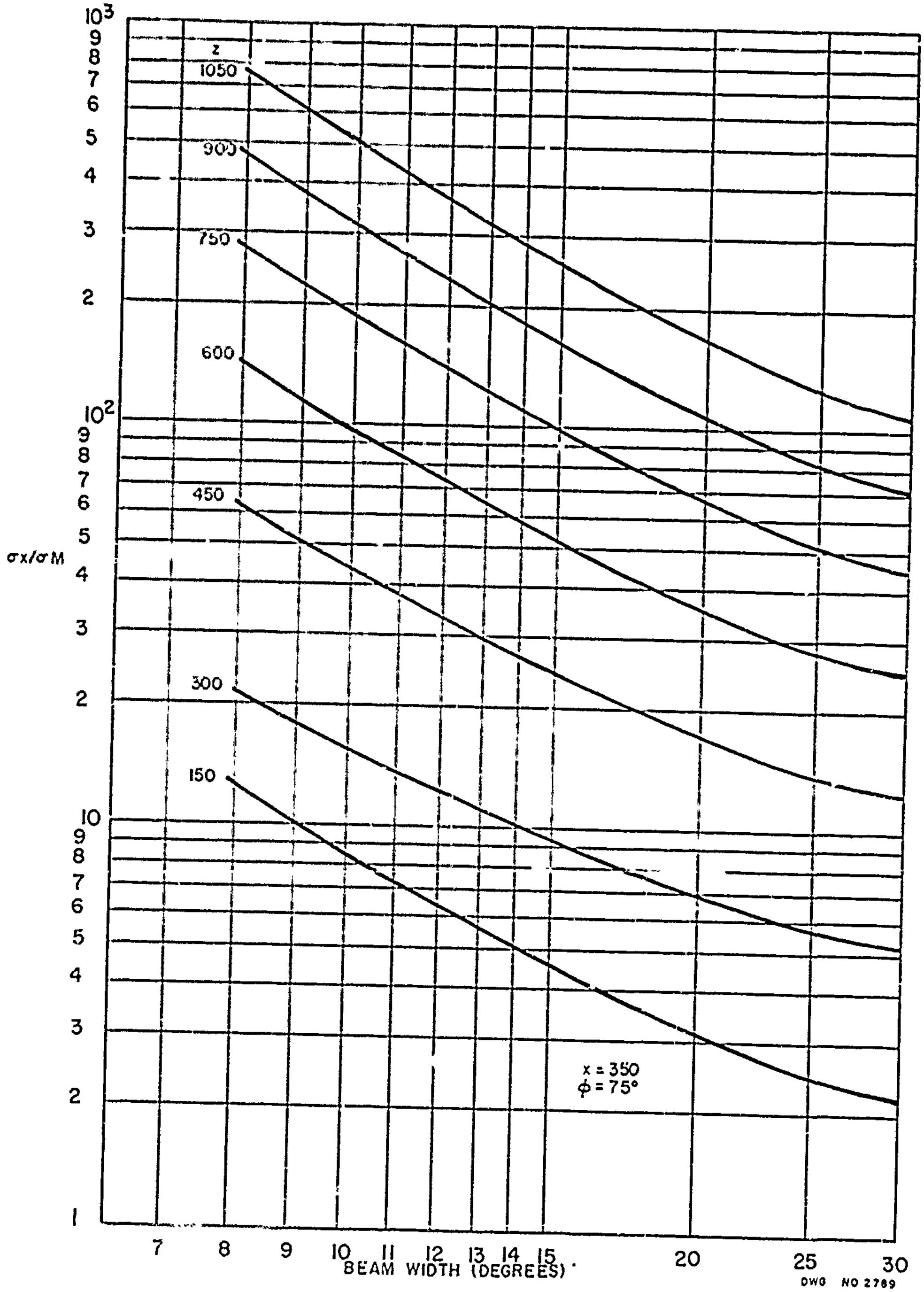
Figure 10c

DWG No 2809



10
9
8
7
6
5
4
3
2
1.0
0.1
.01
σ z/σM
z
150
350
450
600
750
900
1050
x = 350
φ = 45°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2799

Figure 10d



10^3
9
8
7
6
5
4
3
2
10^2
10
1
σx/σM
z
1050
900
750
600
450
300
150
x = 350
φ = 75°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2789

Figure 11a

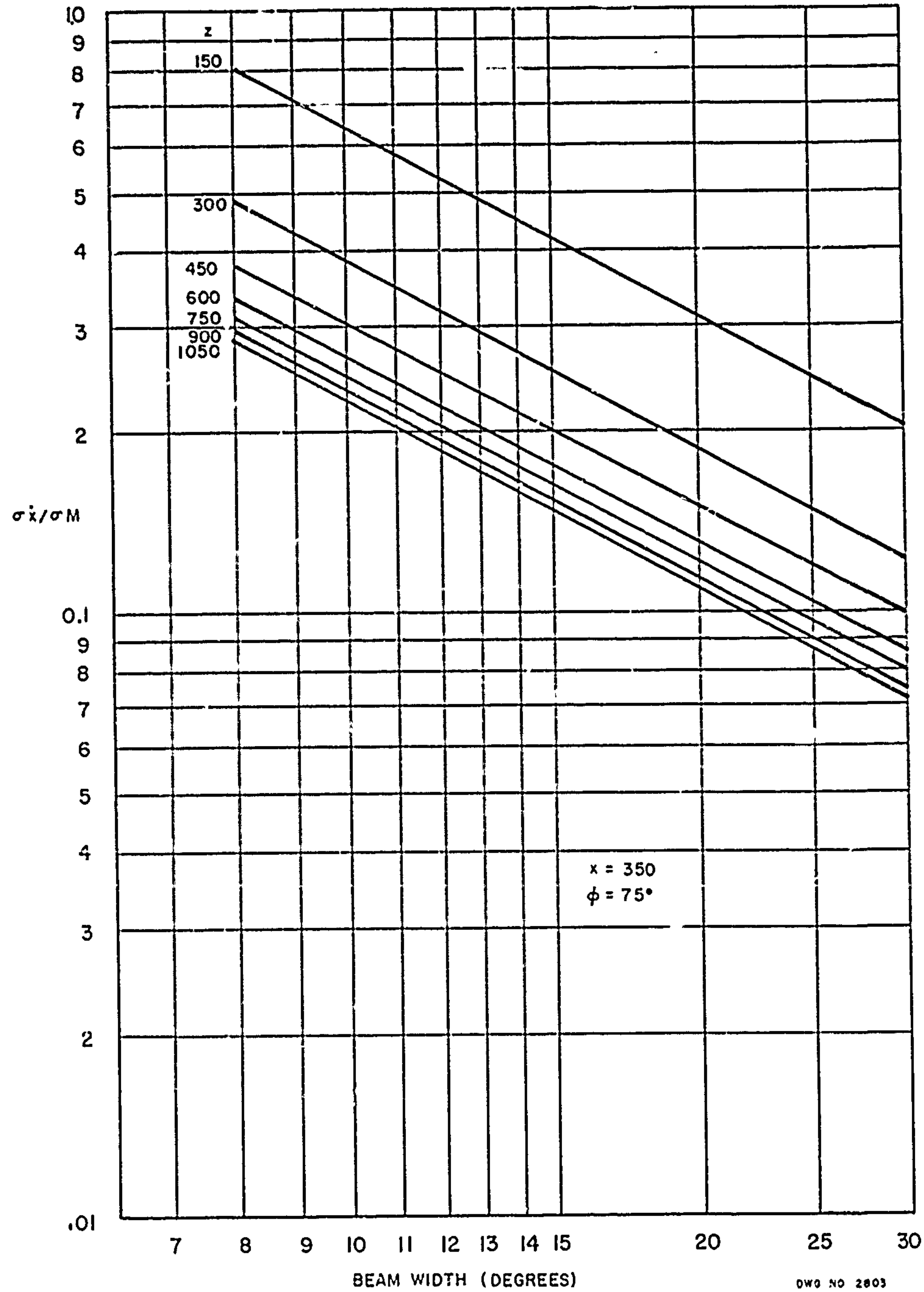

10
9
8
7
6
5
4
3
2
σẋ/σM
0.1
9
8
7
6
5
4
3
2
.01
z
150
300
450
600
750
900
1050
x = 350
φ = 75°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2803

Figure 11c

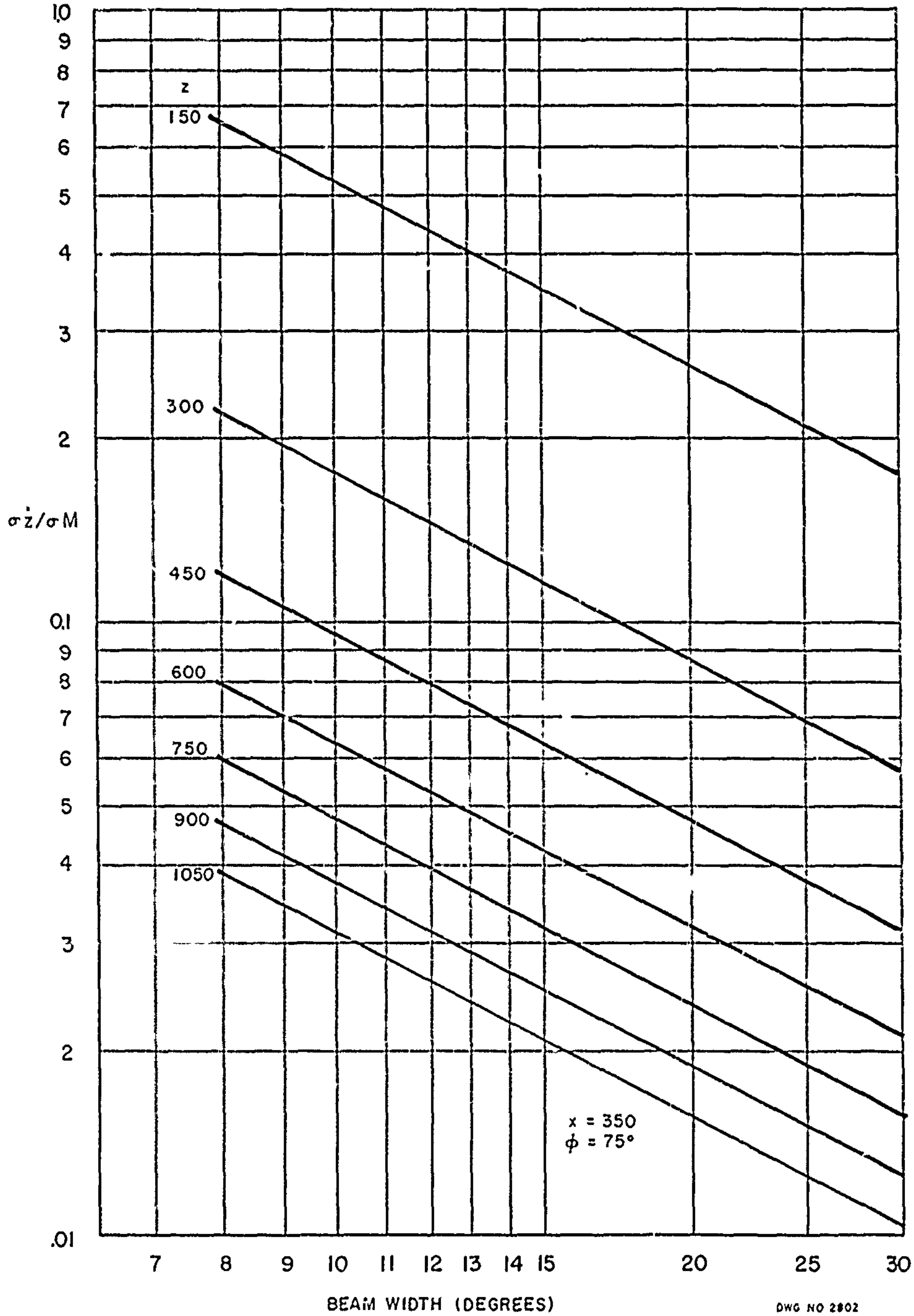
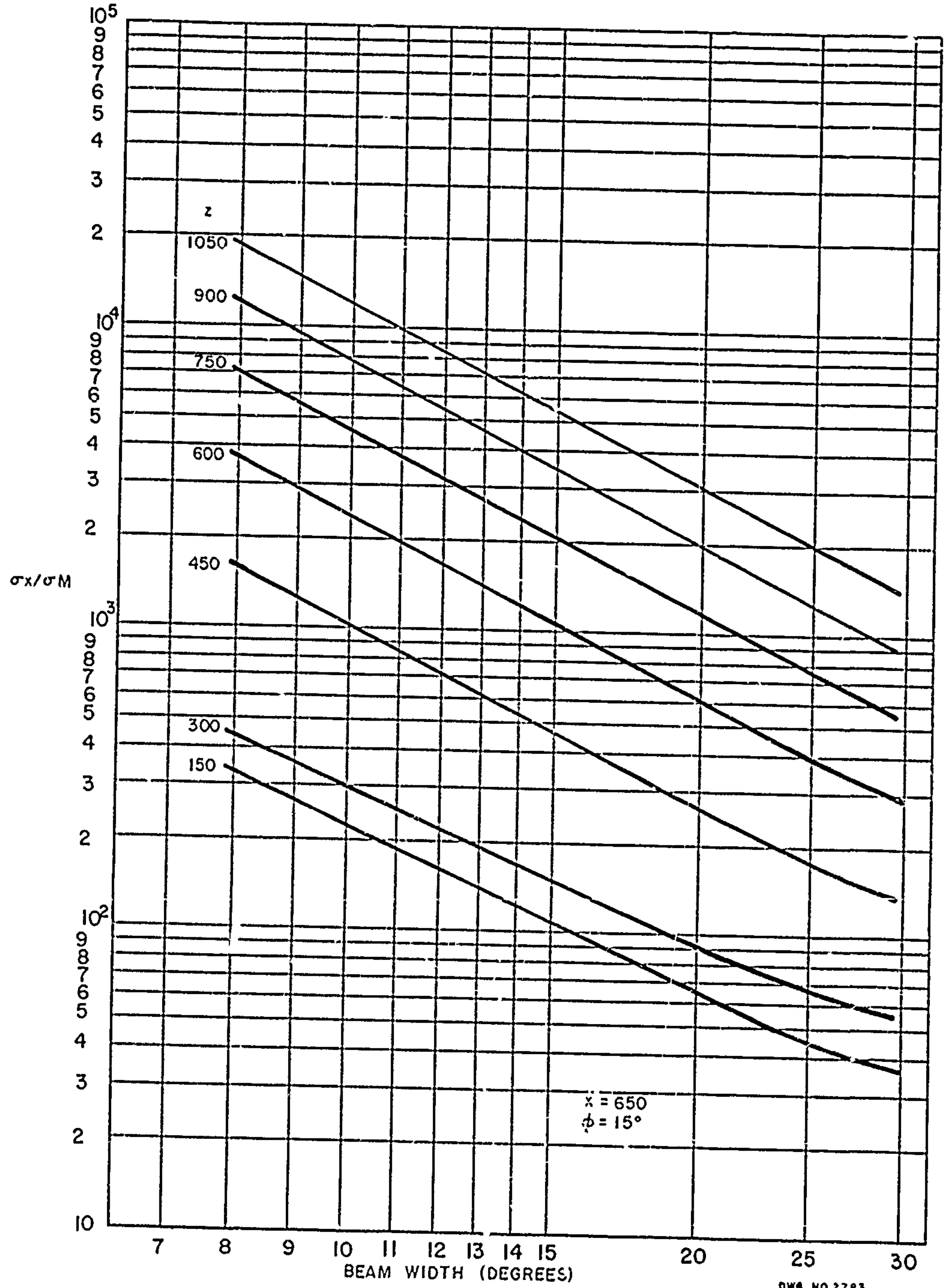

10
9
8
7
6
5
4
3
2
σż/σM
0.1
9
8
7
6
5
4
3
2
.01
z
150
300
450
600
750
900
1050
x = 350
φ = 75°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2802

Figure 11d



$\sigma x/\sigma M$
z
1050
900
750
600
450
300
150
x = 650
φ = 15°
BEAM WIDTH (DEGREES)
DWG NO 2783

Figure 12a

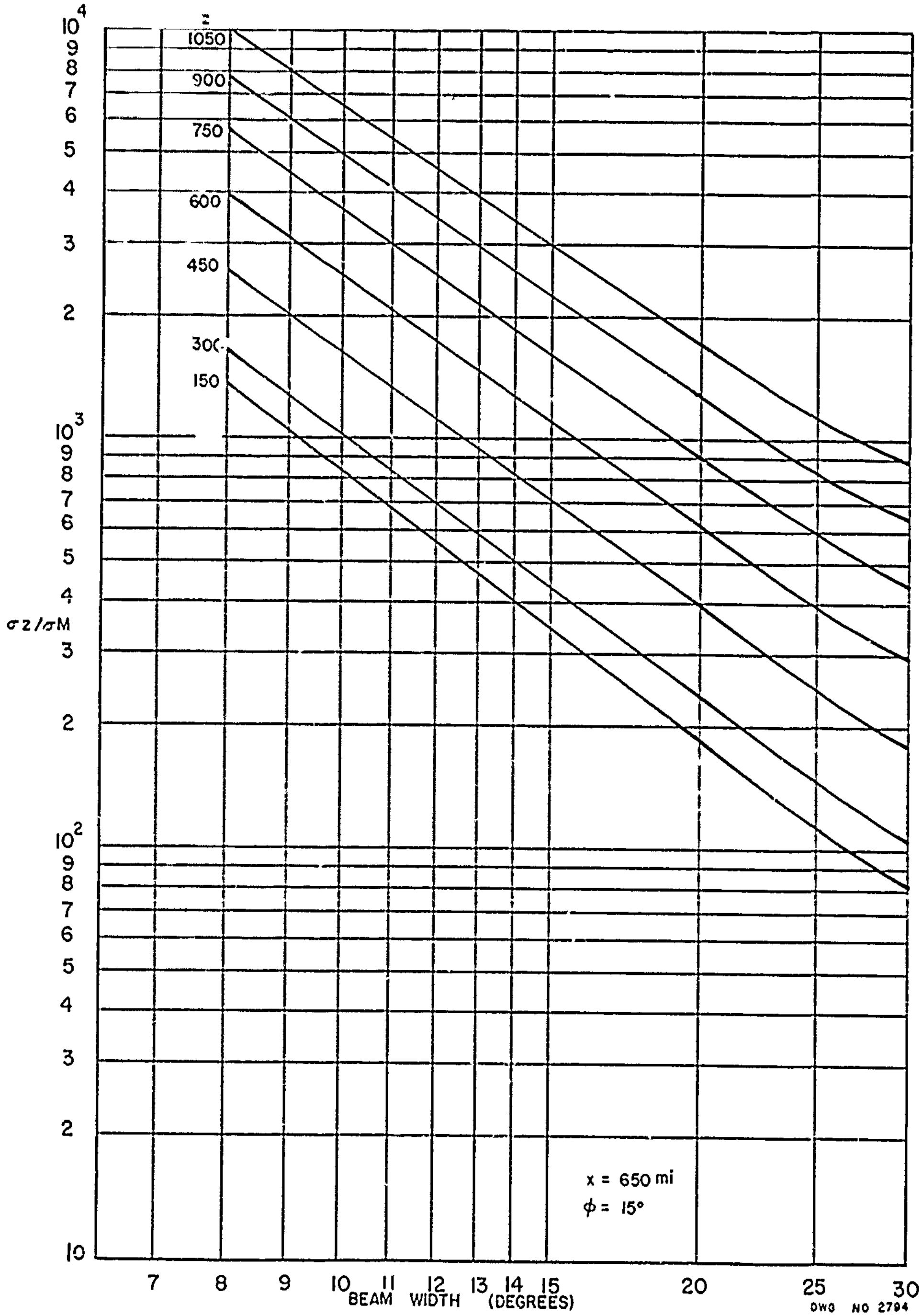

z
1050
900
750
600
450
300
150
10^4
9
8
7
6
5
4
3
2
10^3
9
8
7
6
5
4
3
2
10^2
9
8
7
6
5
4
3
2
10
$\sigma z / \sigma M$
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
x = 650 mi
ϕ = 15°
DWG NO 2794

Figure 12b

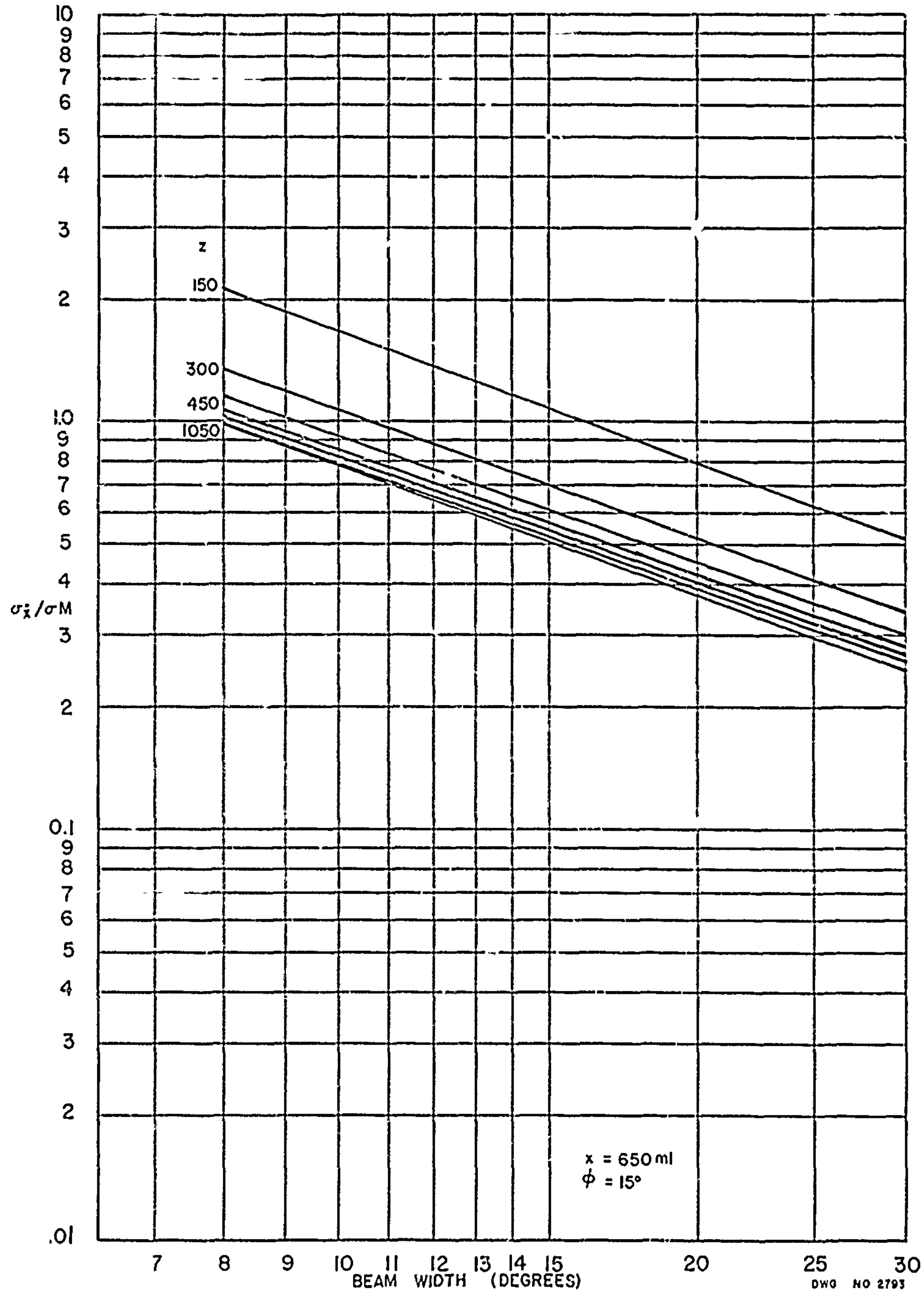

10
9
8
7
6
5
4
3
2
1.0
0.1
.01
z
150
300
450
1050
$\sigma_{\dot{x}}/\sigma M$
x = 650 mi
ϕ = 15°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2793

Figure 12c

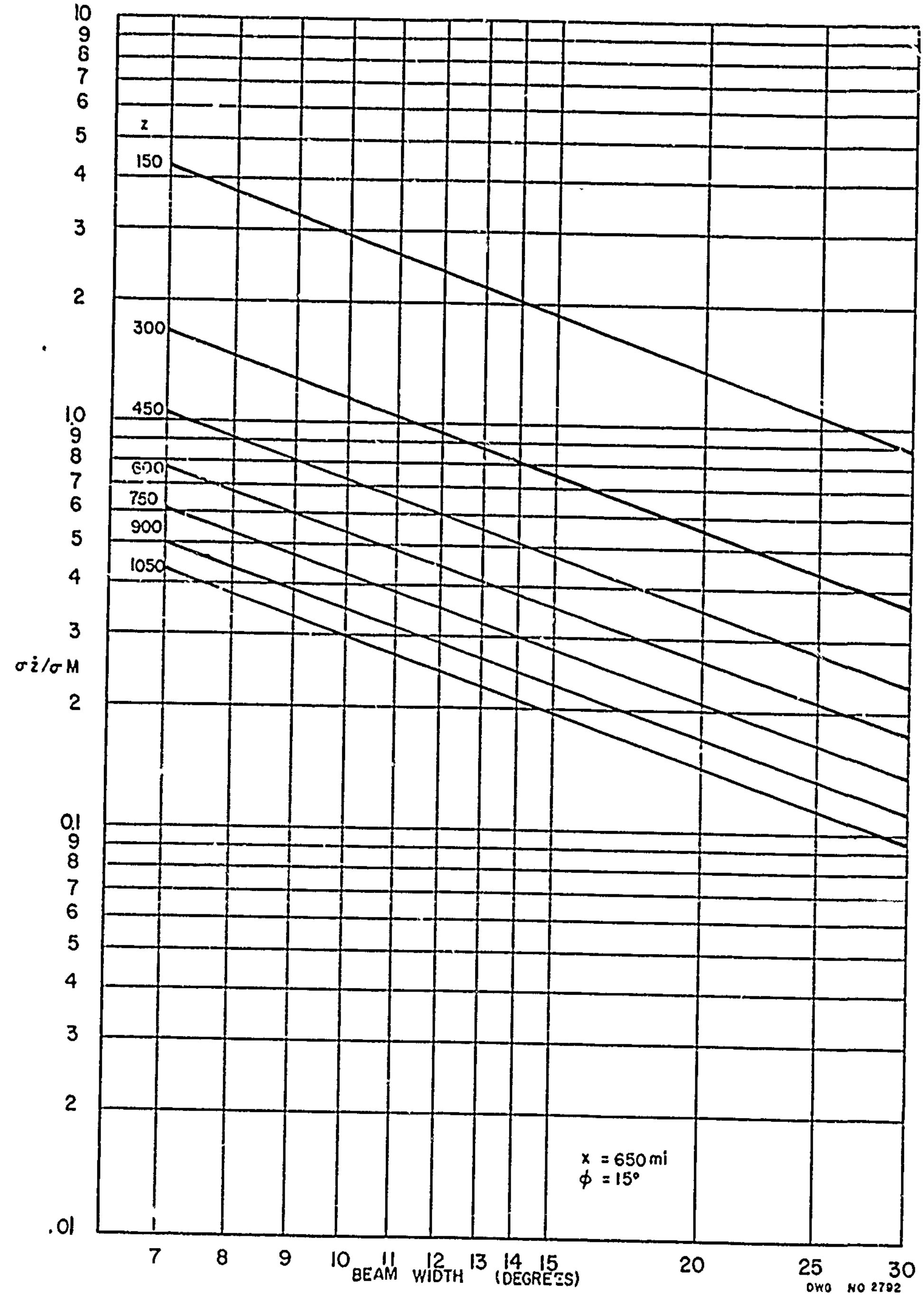

10
9
8
7
6
5
4
3
2
1.0
9
8
7
6
5
4
3
2
0.1
9
8
7
6
5
4
3
2
.01
z
150
300
450
600
750
900
1050
σż/σM
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
x = 650 mi
ϕ = 15°
DWG NO 2792

Figure 12d

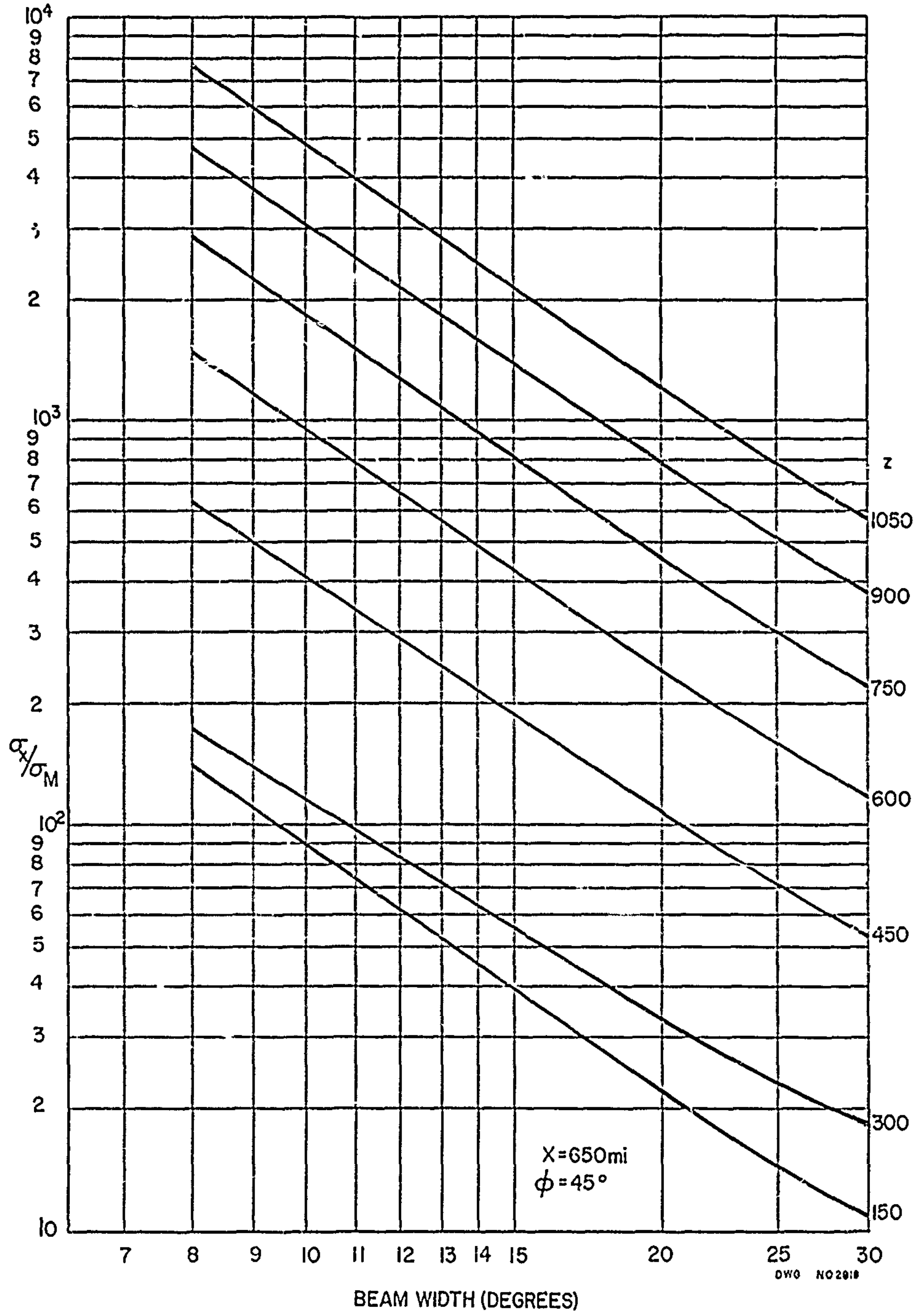
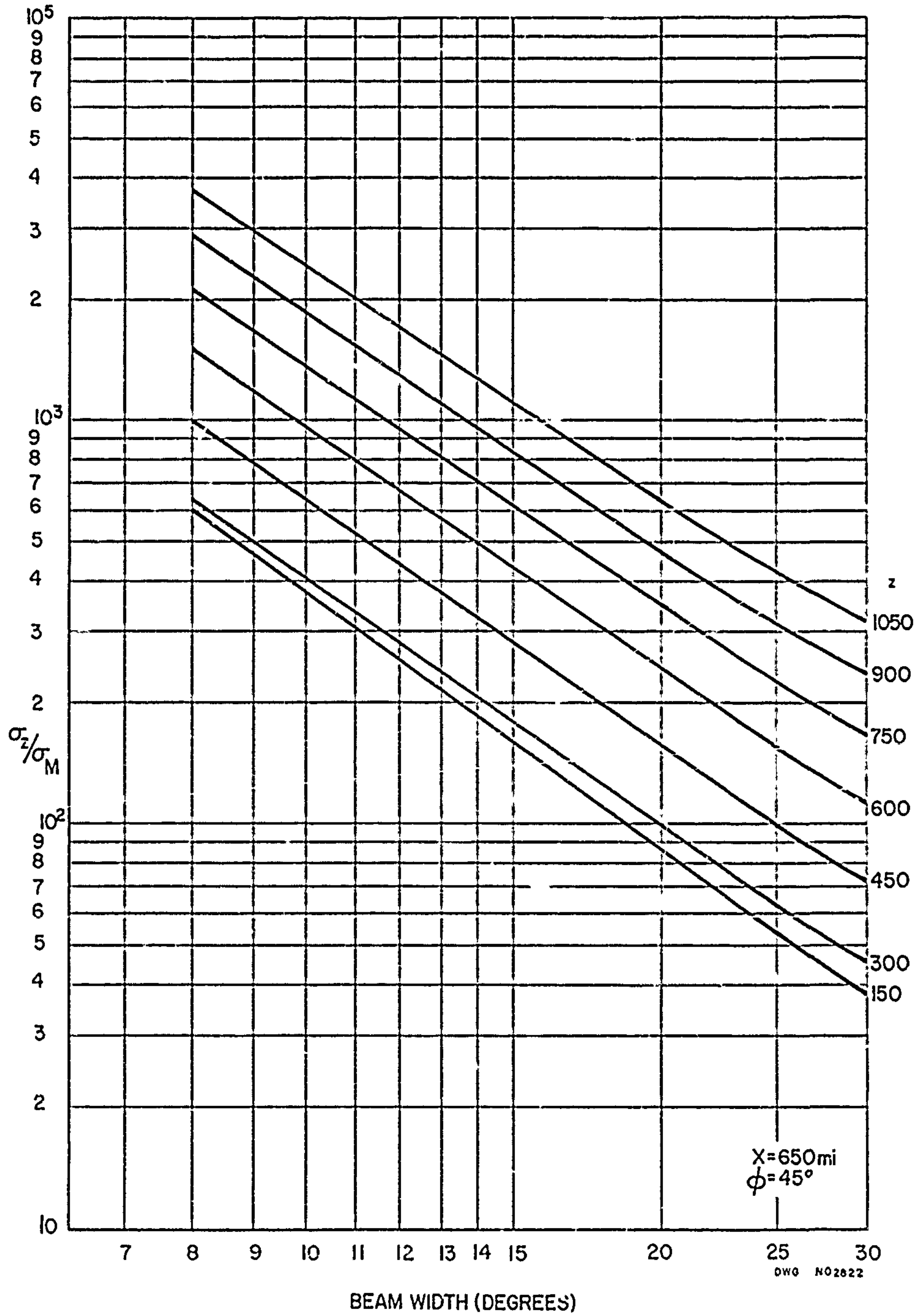

10^4
9
8
7
6
5
4
3
2
10^3
9
8
7
6
5
4
3
2
σx/σM
10^2
9
8
7
6
5
4
3
2
10
Z
1050
900
750
600
450
300
150
X=650mi
φ=45°
7
8
9
10
11
12
13
14
15
20
25
30
DWG NO2919
BEAM WIDTH (DEGREES)

Figure 13a



σ_z/σ_M
10
2
3
4
5
6
7
8
9
10^2
2
3
4
5
6
7
8
9
10^3
2
3
4
5
6
7
8
9
10^5
7
8
9
10
11
12
13
14
15
20
25
30
z
1050
900
750
600
450
300
150
X=650mi
ϕ=45°
DWG NO2822
BEAM WIDTH (DEGREES)

Figure 13b

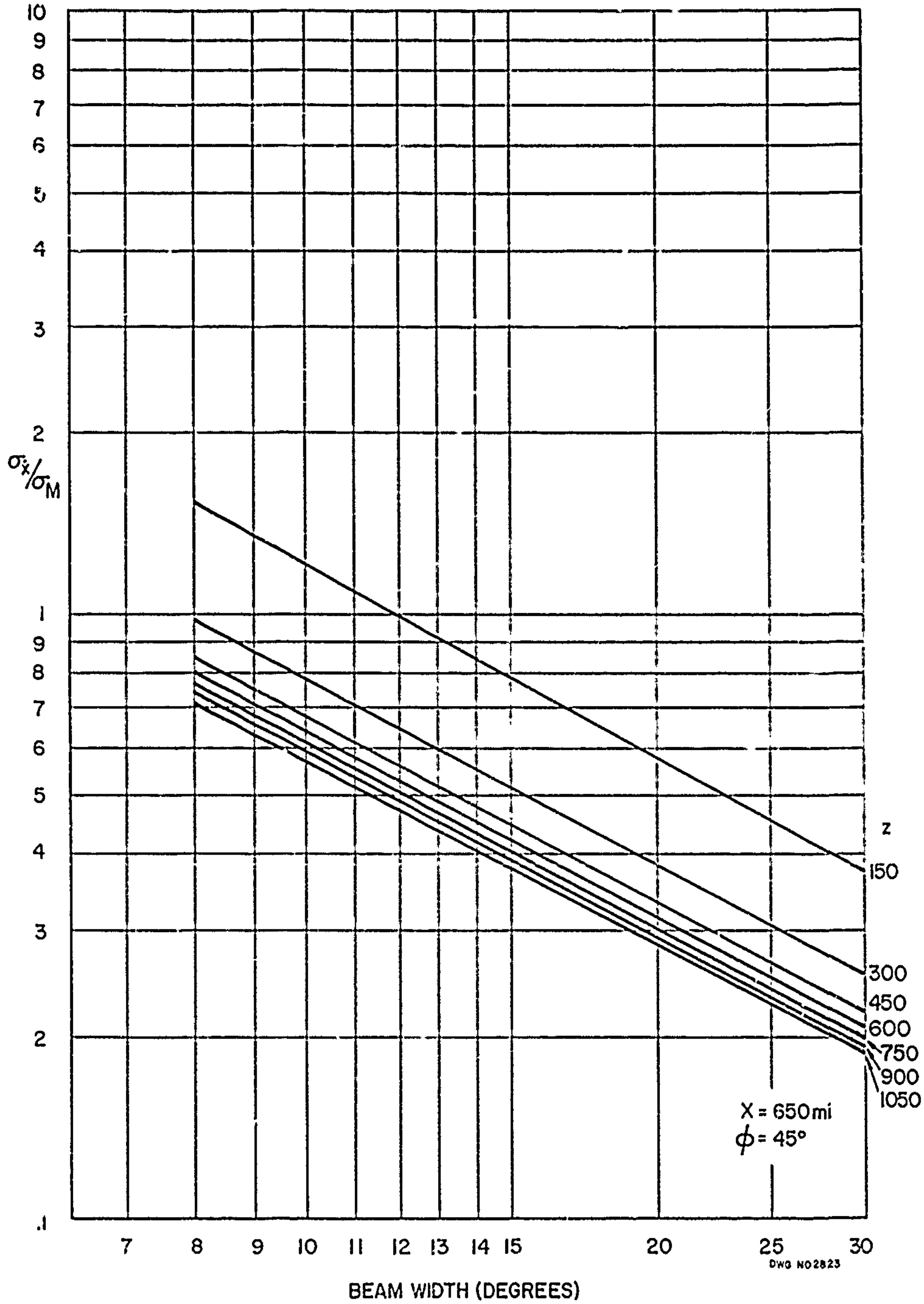
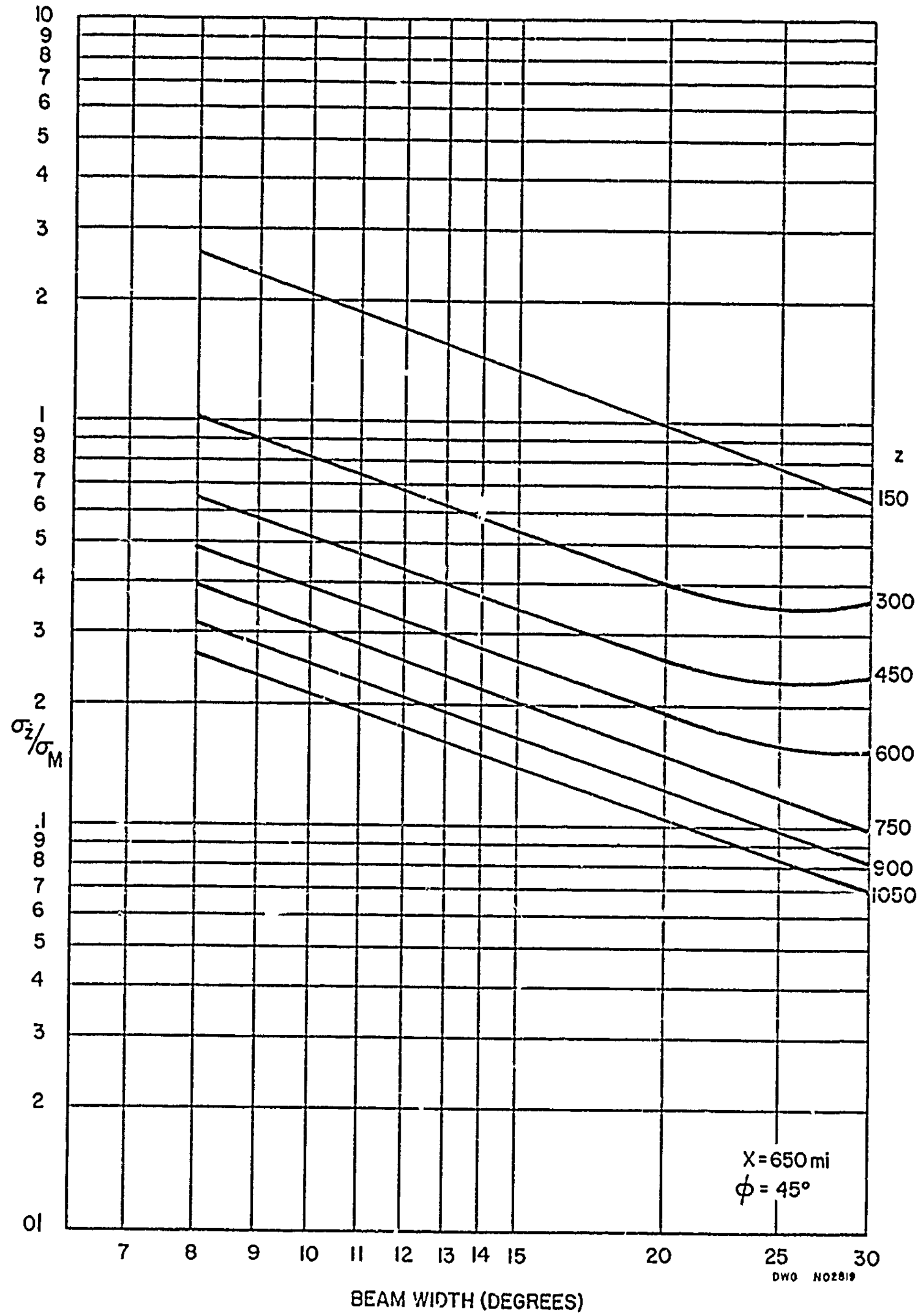
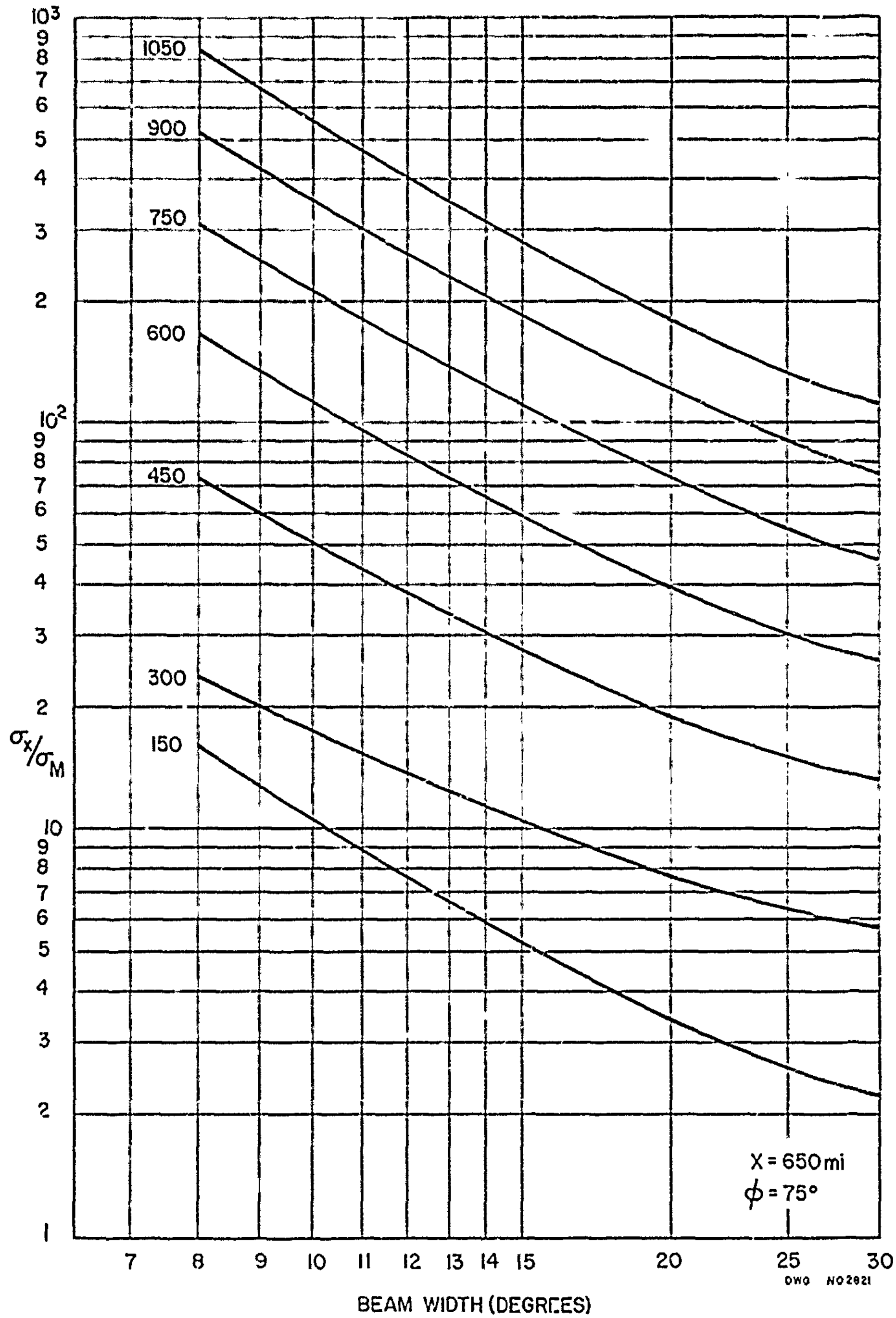

10
9
8
7
6
5
4
3
2
$\sigma_{\dot{x}}/\sigma_M$
1
9
8
7
6
5
4
3
2
.1
7
8
9
10
11
12
13
14
15
20
25
30
DWG NO2823
BEAM WIDTH (DEGREES)
Z
150
300
450
600
750
900
1050
X = 650mi
ϕ = 45°

Figure 13c



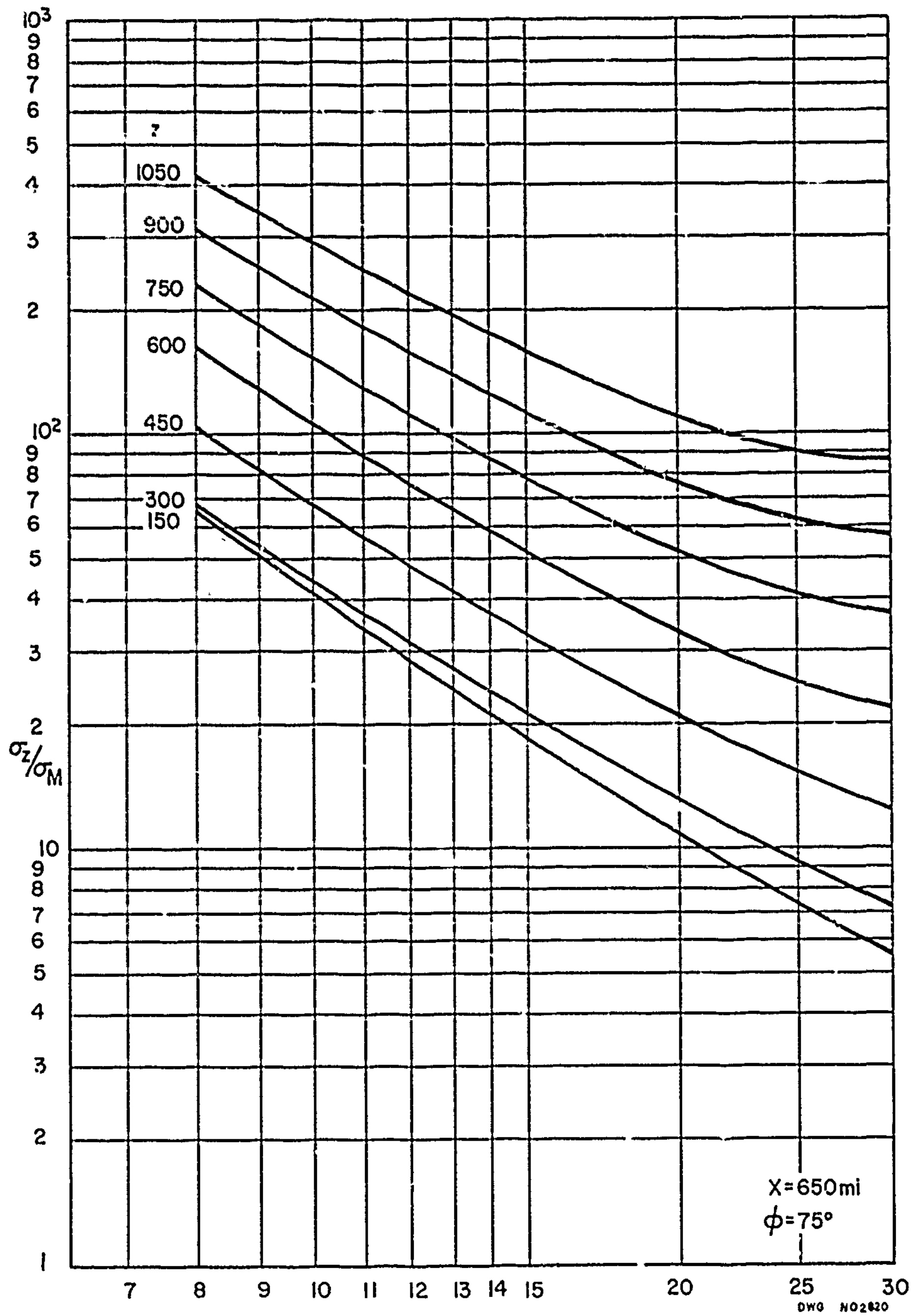
10
9
8
7
6
5
4
3
2
1
.1
01
$\sigma_{\dot{z}}/\sigma_M$
z
150
300
450
600
750
900
1050
X = 650 mi
ϕ = 45°
7
8
9
10
11
12
13
14
15
20
25
30
DWG NO2819
BEAM WIDTH (DEGREES)

Figure 13d



PHILCO

GOVERNMENT & INDUSTRIAL DIVISION
WESTERN DEVELOPMENT LABORATORIES



10^3
9
8
7
6
5
4
3
2
10^2
10
1
τ
1050
900
750
600
450
300
150
σ_z/σ_M
7
8
9
10
11
12
13
14
15
20
25
30
DWG NO2820
BEAM WIDTH (DEGREES)
X=650mi
ϕ=75°

Figure 14b

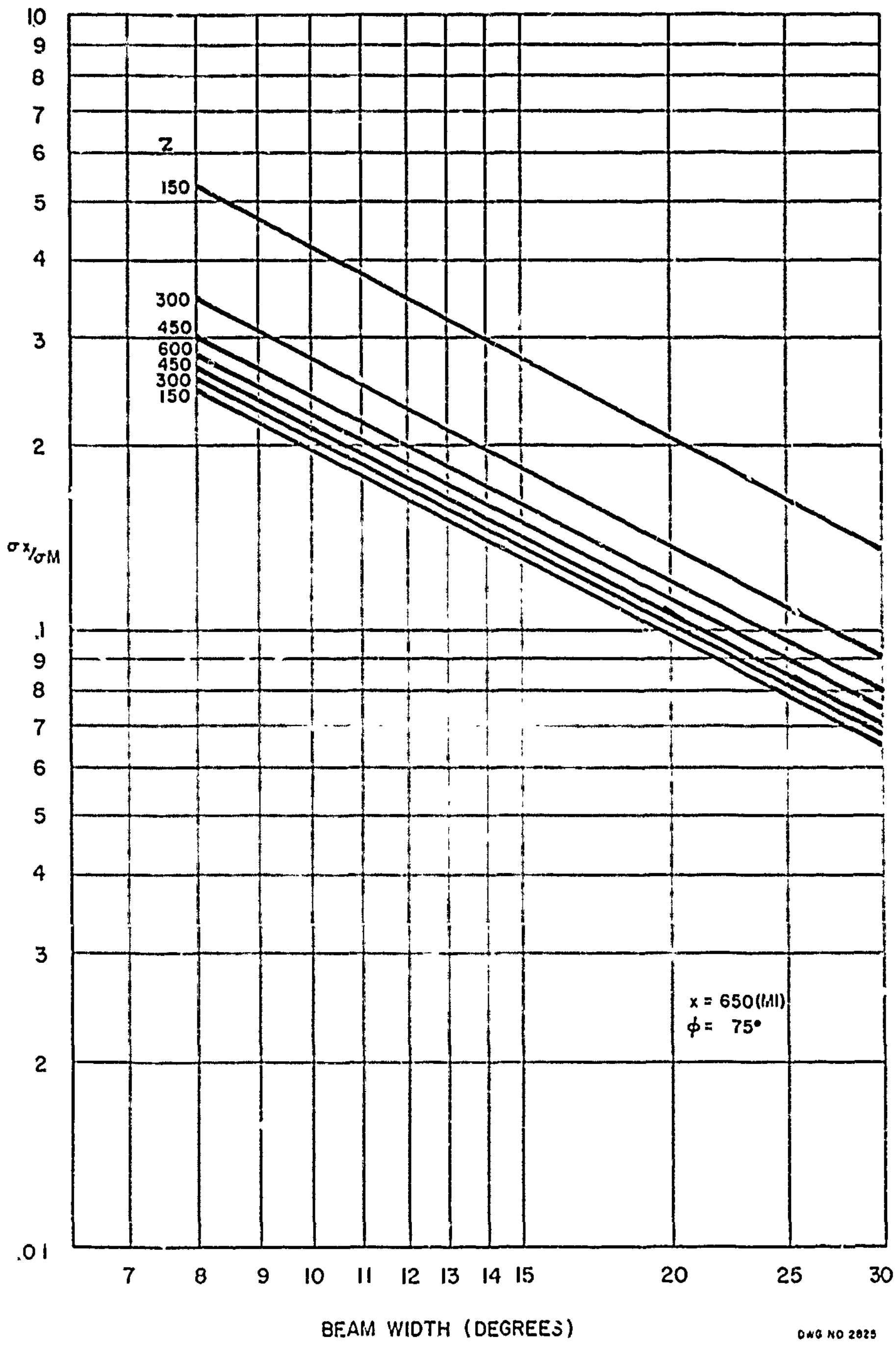

10
9
8
7
6
5
4
3
2
$\sigma x / \sigma M$
.1
9
8
7
6
5
4
3
2
.01
Z
150
300
450
600
450
300
150
x = 650(MI)
ϕ = 75°
7
8
9
10
11
12
13
14
15
20
25
30
BEAM WIDTH (DEGREES)
DWG NO 2825

Figure 14c

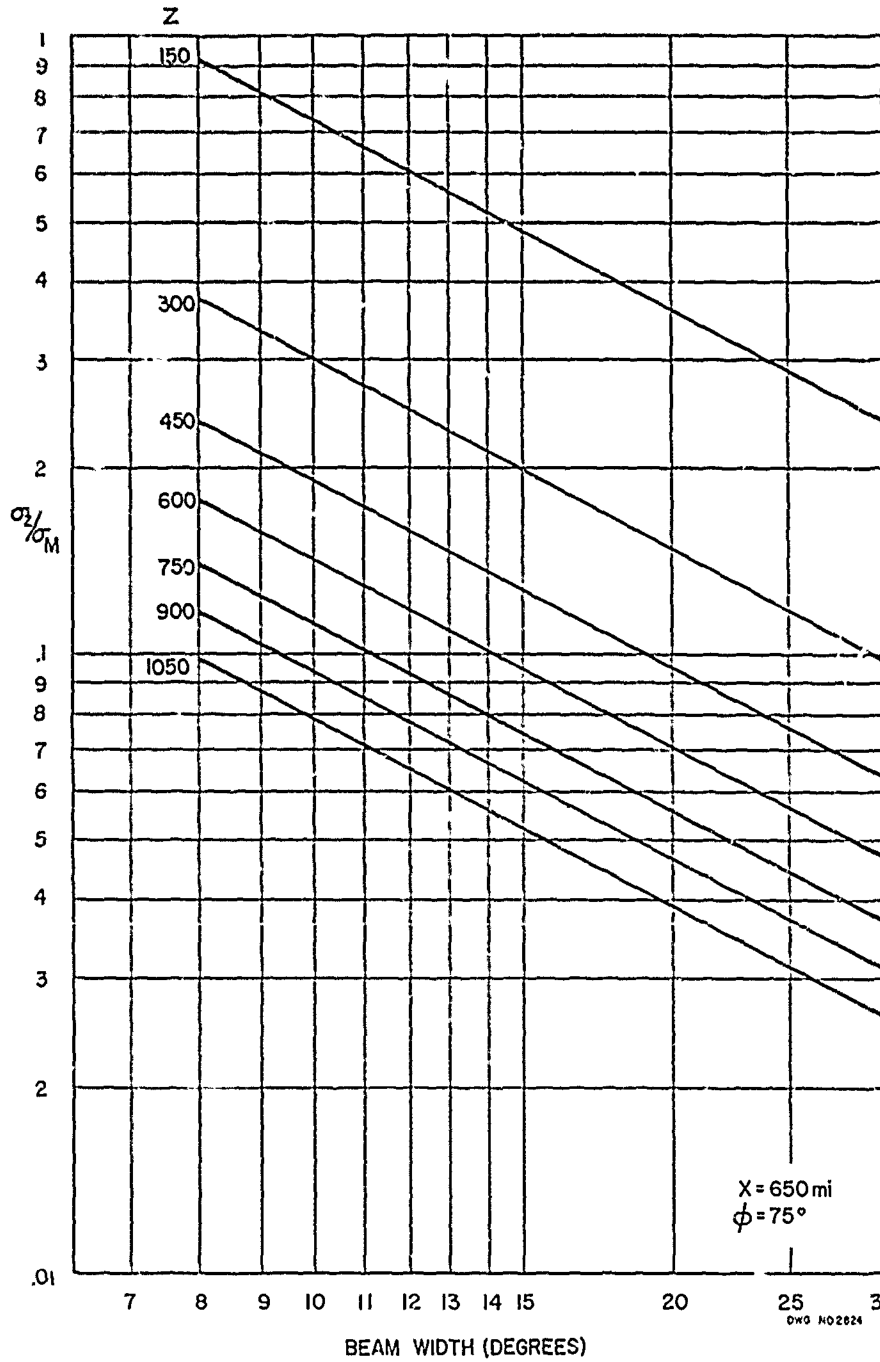

Z
150
300
450
600
750
900
1050
1
9
8
7
6
5
4
3
2
.1
9
8
7
6
5
4
3
2
.01
σ_z/σ_M
7
8
9
10
11
12
13
14
15
20
25
30
X = 650 mi
ϕ = 75°
DWG NO2824
BEAM WIDTH (DEGREES)

Figure 14d

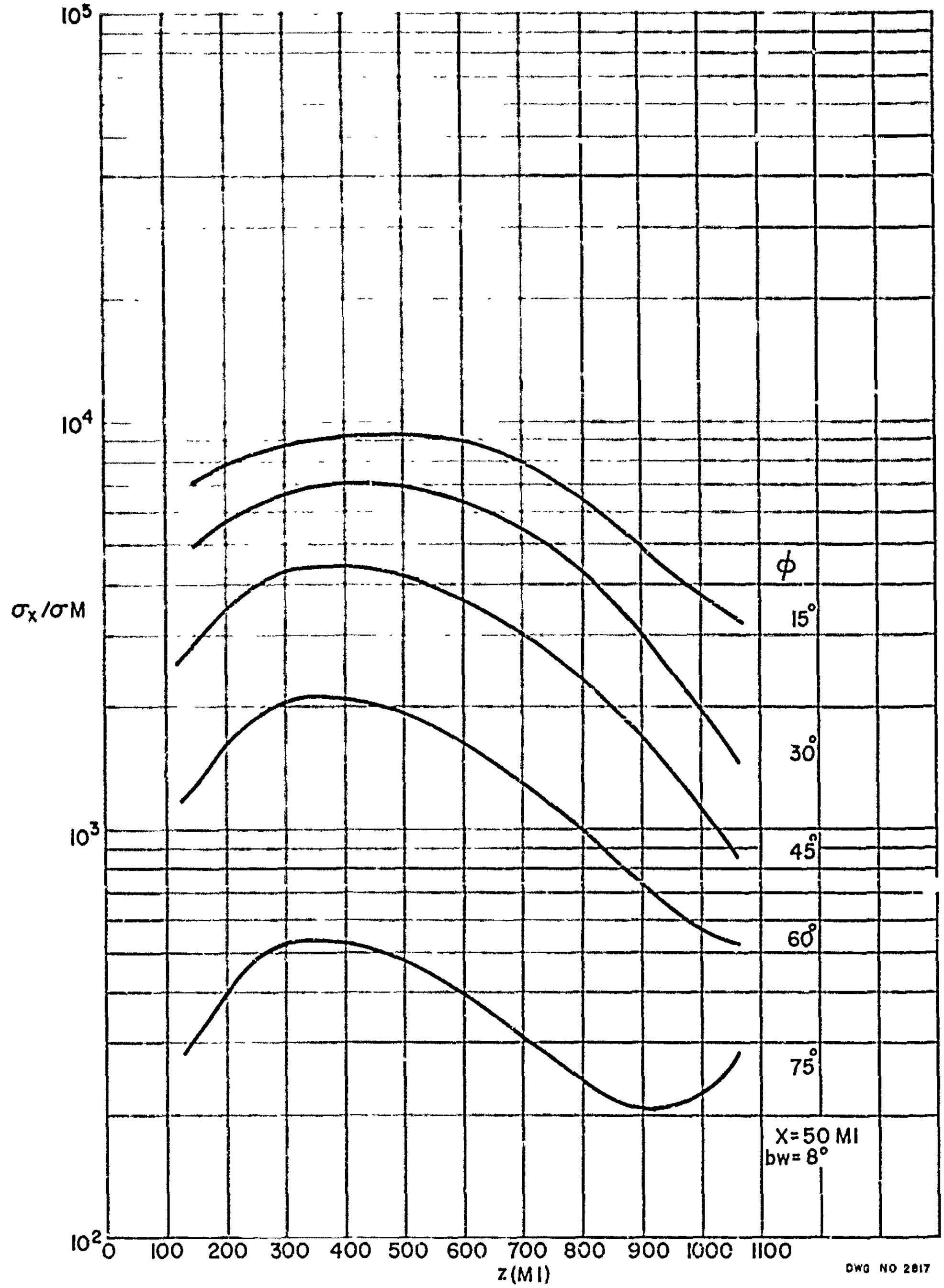

10^5
10^4
10^3
10^2
σx/σM
φ
15°
30°
45°
60°
75°
X=50 MI
bw= 8°
0
100
200
300
400
500
600
700
800
900
1000
1100
z(MI)
DWG NO 2817

Figure [illegible]a

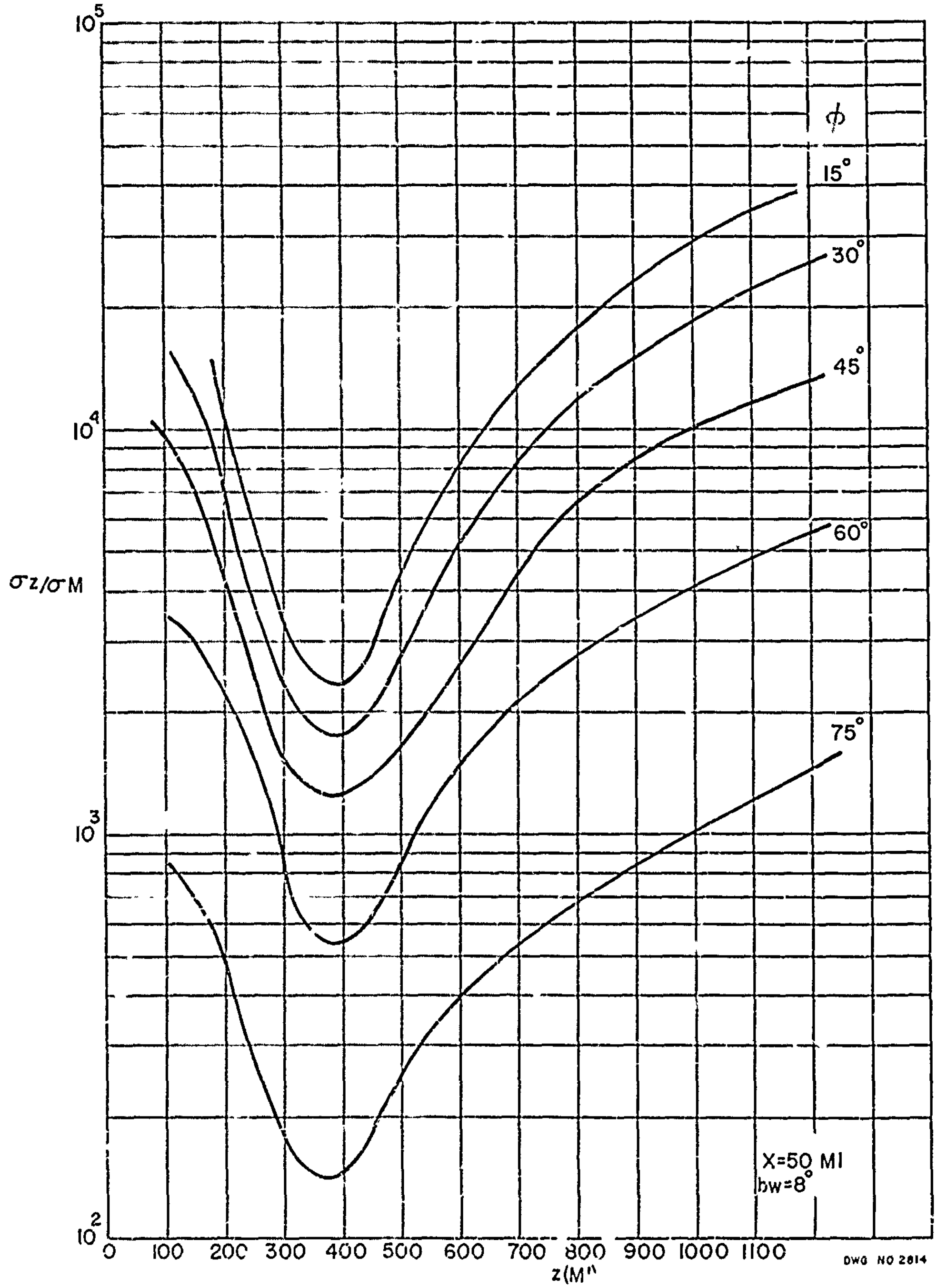

10^5
10^4
10^3
10^2
σz/σM
φ
15°
30°
45°
60°
75°
X=50 MI
bw=8°
0 100 200 300 400 500 600 700 800 900 1000 1100
z(M)
DWG NO 2814

Figure 15o

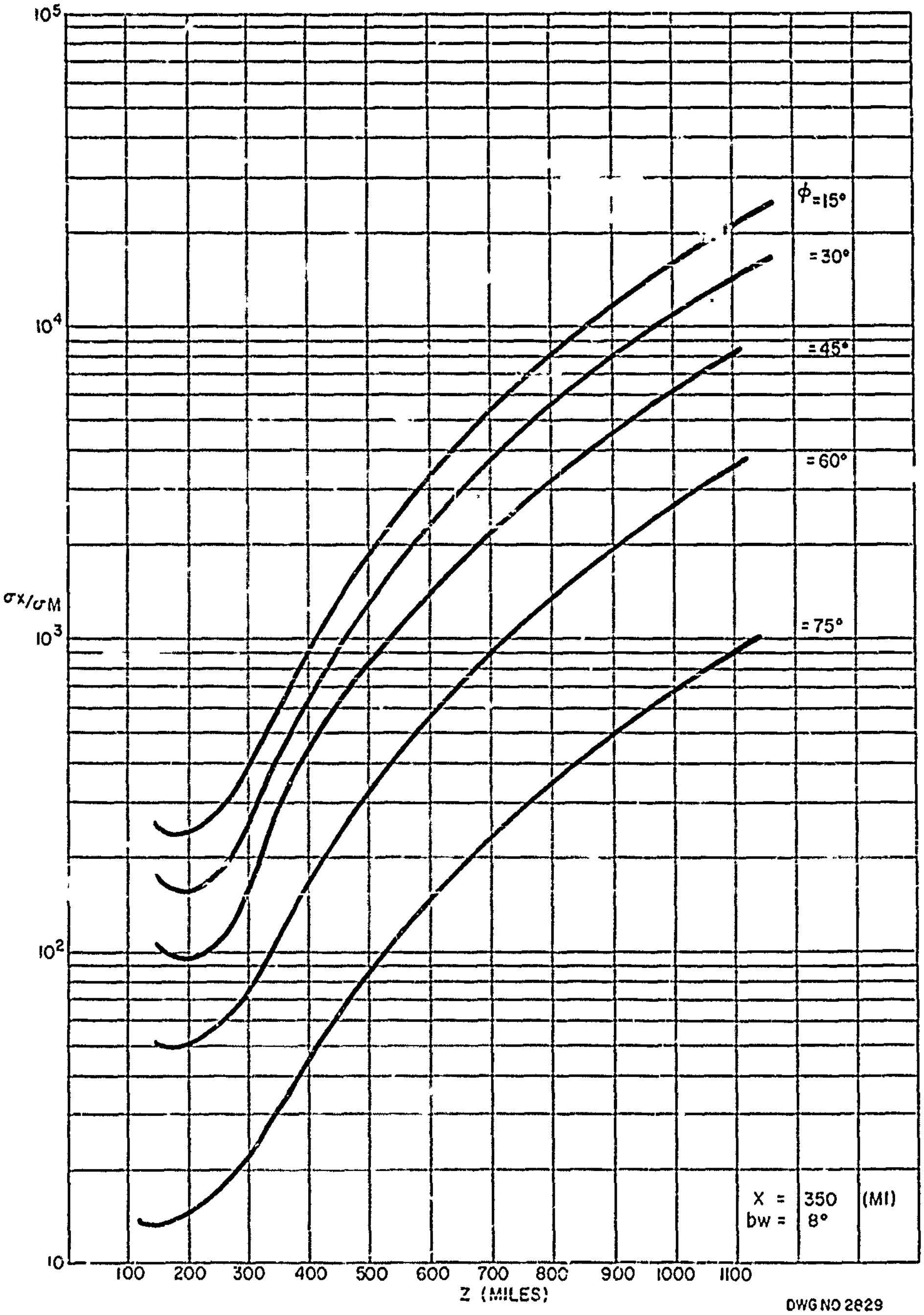

10^5
10^4
10^3
10^2
10
$\sigma x/\sigma M$
ϕ=15°
=30°
=45°
=60°
=75°
X = 350 (MI)
bw = 8°
100 200 300 400 500 600 700 800 900 1000 1100
Z (MILES)
DWG NO 2829

Figure 16a

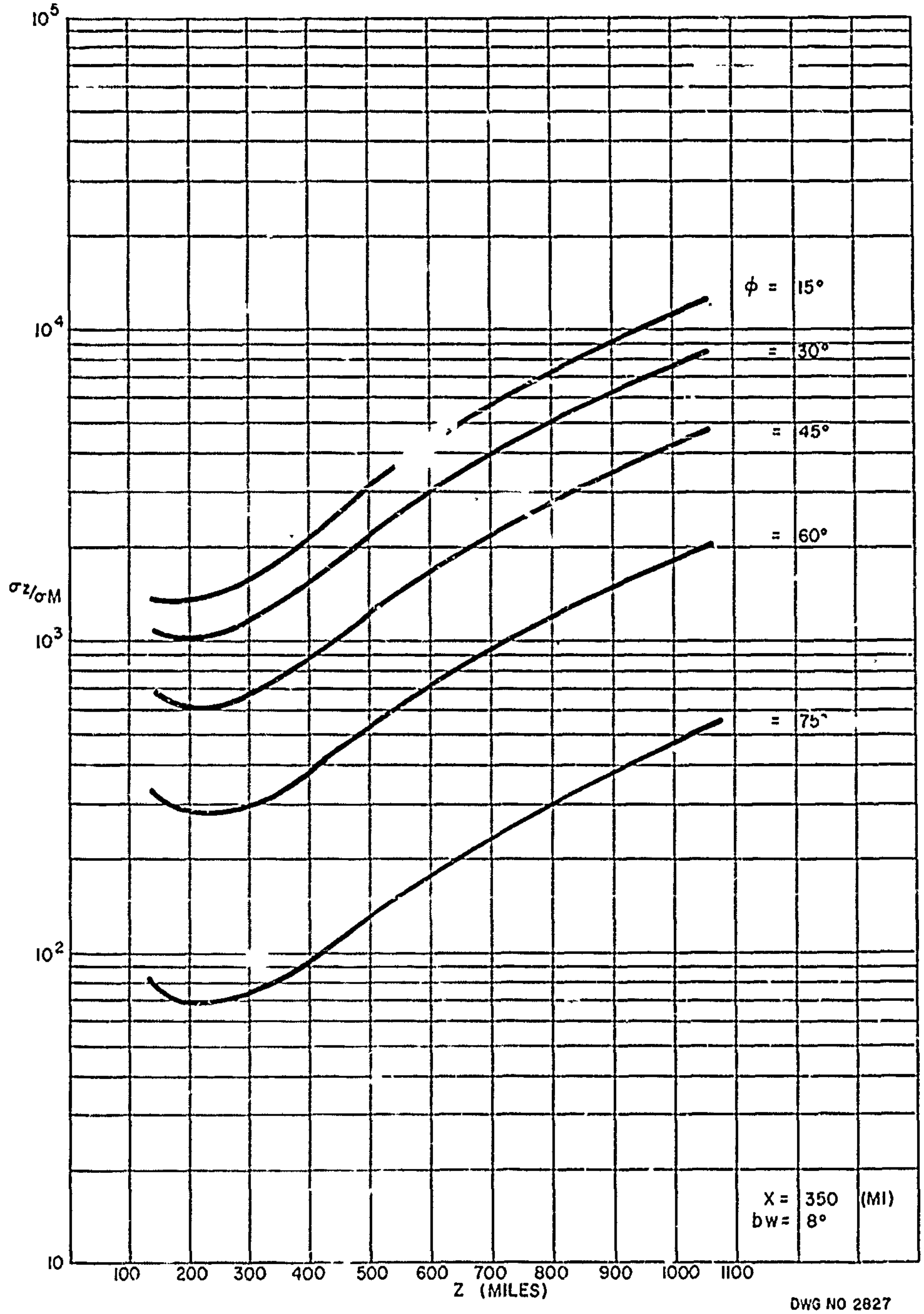
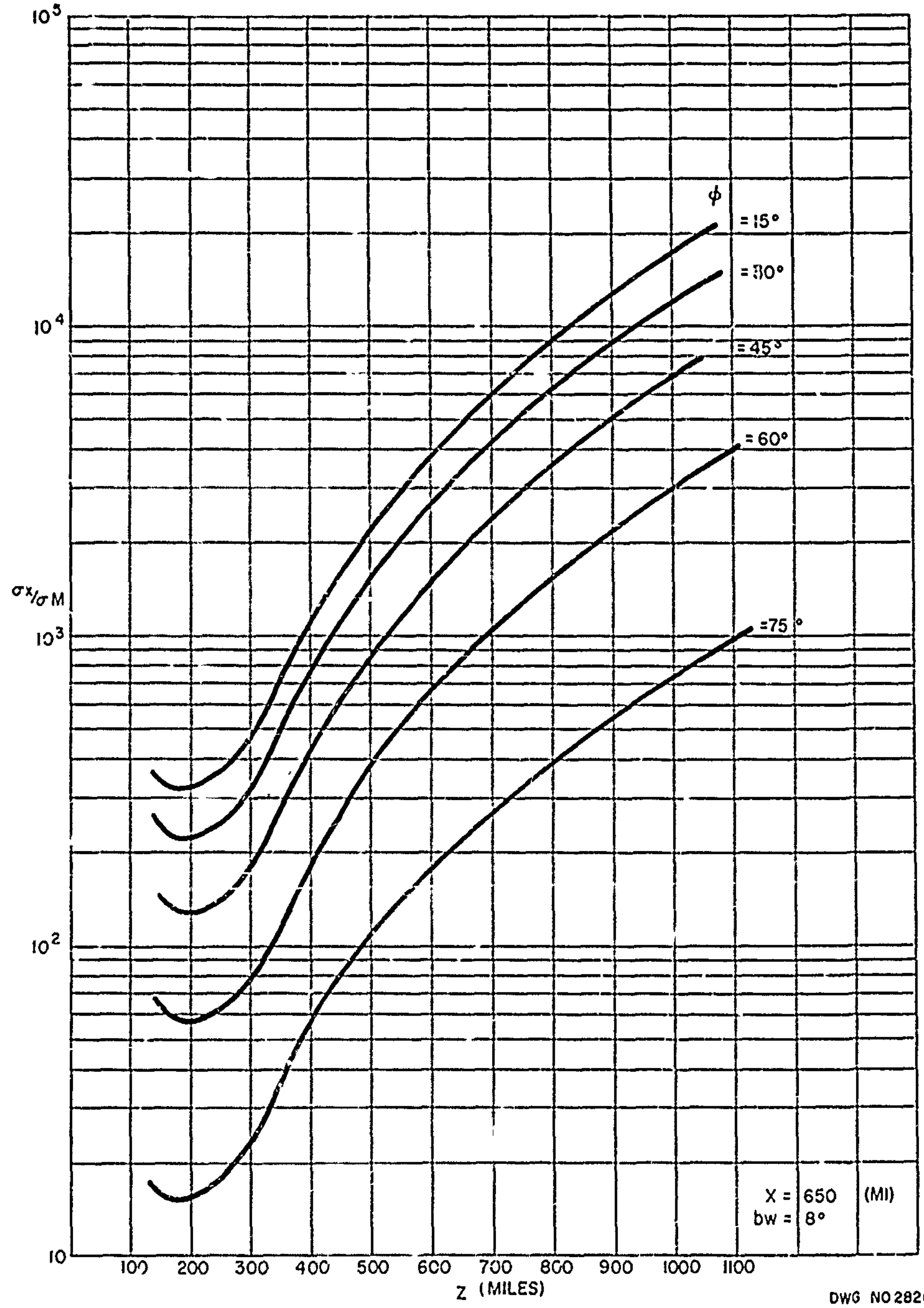

10^5
10^4
10^3
10^2
10
σ_Z/σ_M
ϕ = 15°
= 30°
= 45°
= 60°
= 75°
X = 350 (MI)
bw = 8°
100 200 300 400 500 600 700 800 900 1000 1100
Z (MILES)
DWG NO 2827

Figure 16b



DWG NO 2826

Figure 17a

PHILCO

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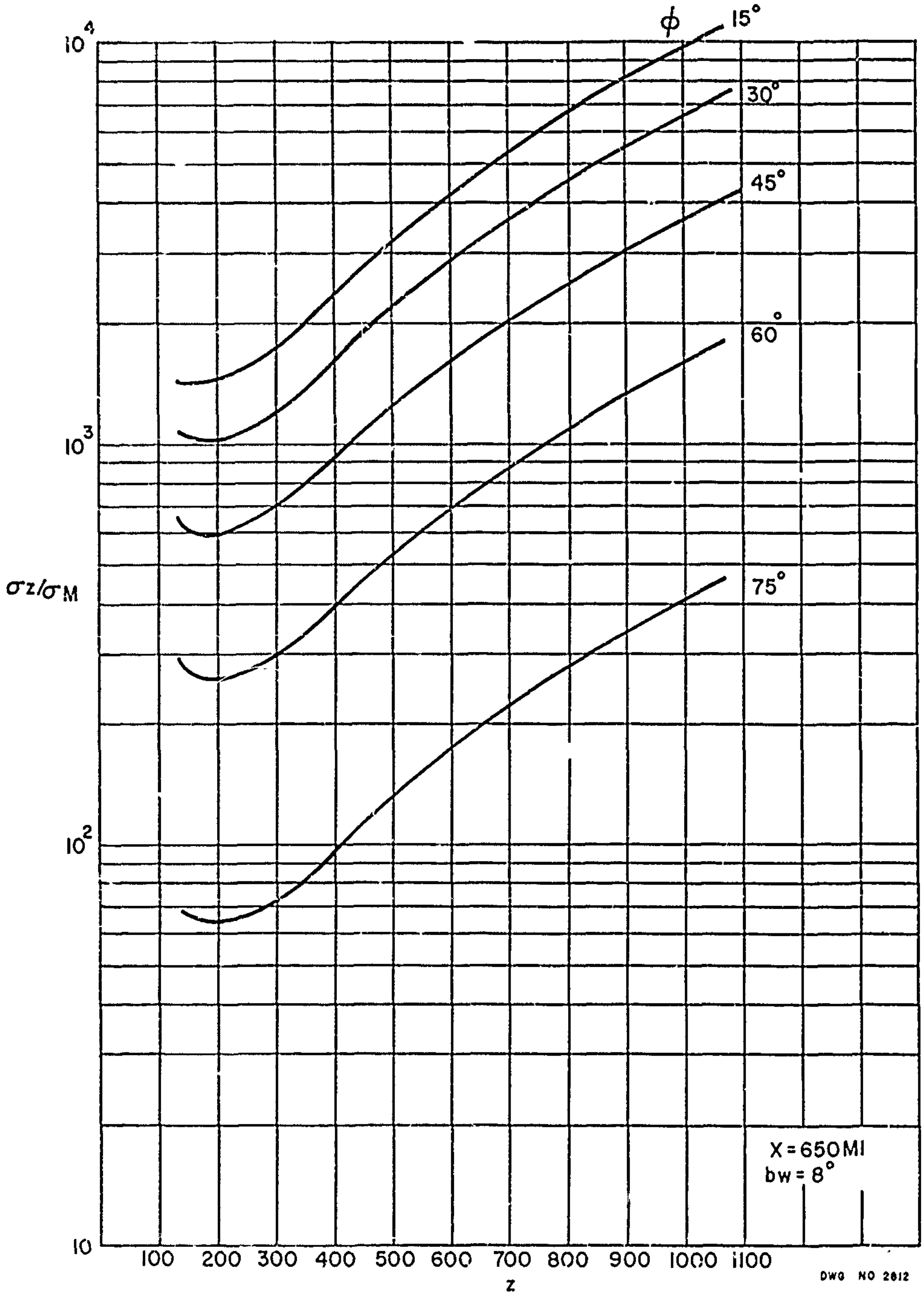
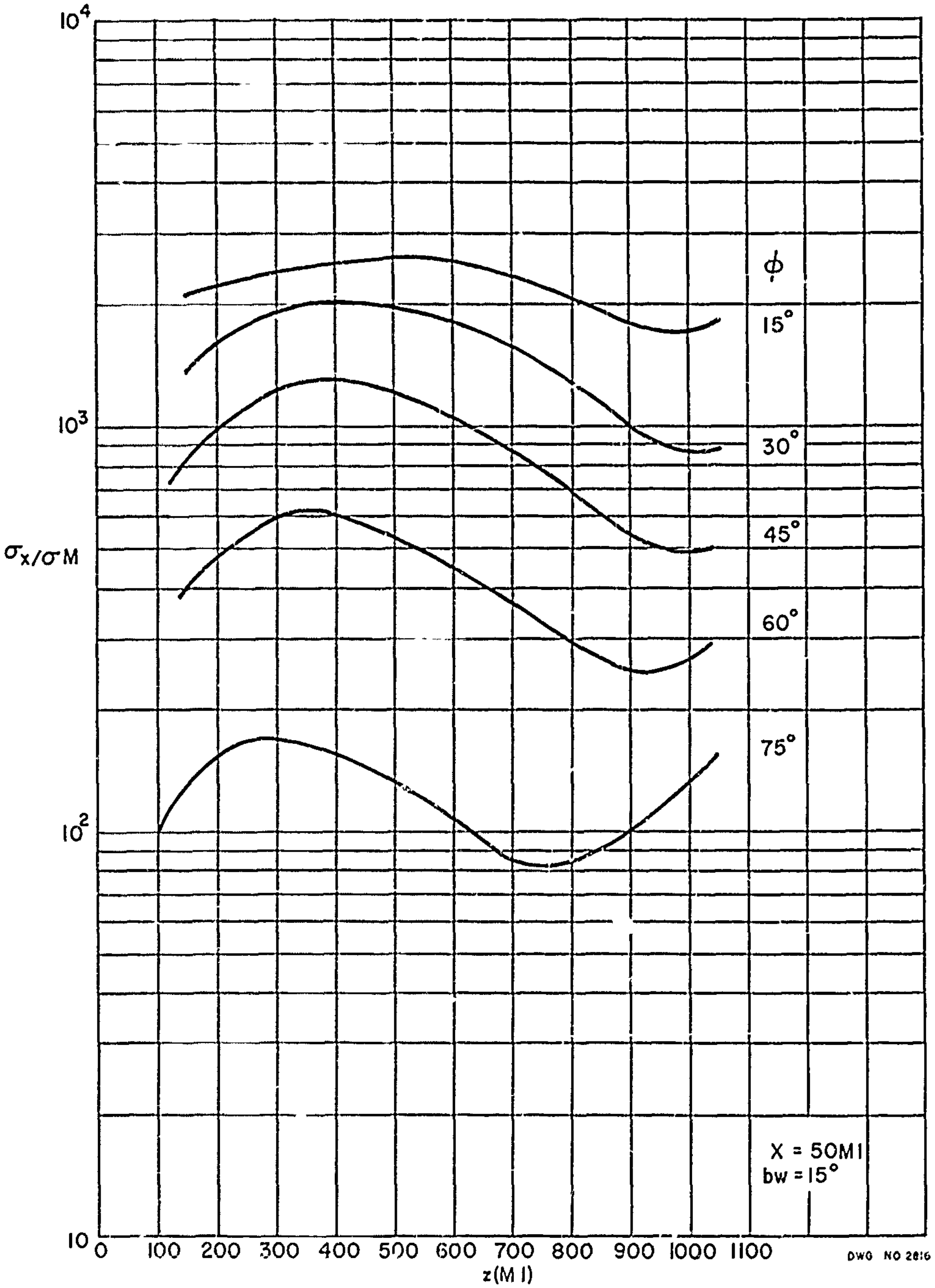

ϕ
15°
30°
45°
60°
75°
10^4
10^3
10^2
10
$\sigma z/\sigma_M$
X = 650 MI
bw = 8°
100 200 300 400 500 600 700 800 900 1000 1100
z
DWG NO 2812

Figure 17b



10^4
10^3
10^2
10
$\sigma_x/\sigma M$
ϕ
15°
30°
45°
60°
75°
X = 50MI
bw = 15°
0 100 200 300 400 500 600 700 800 900 1000 1100
z(MI)
DWG NO 2816

Figure 18a

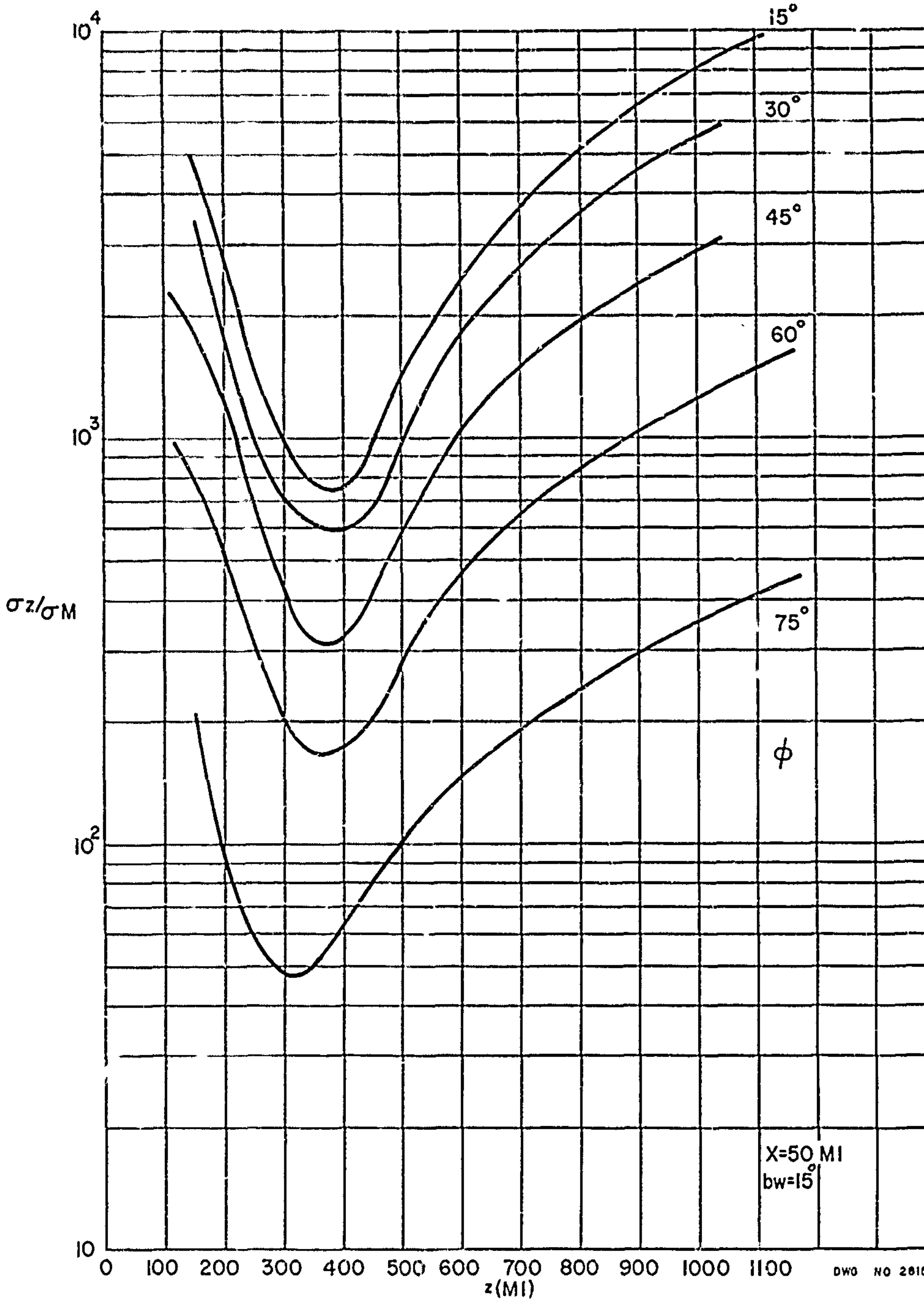

10^4
10^3
10^2
10
$\sigma z/\sigma M$
15°
30°
45°
60°
75°
ϕ
X=50 MI
bw=15°
0
100
200
300
400
500
600
700
800
900
1000
1100
z(MI)
DWG NO 2616

Figure 18b

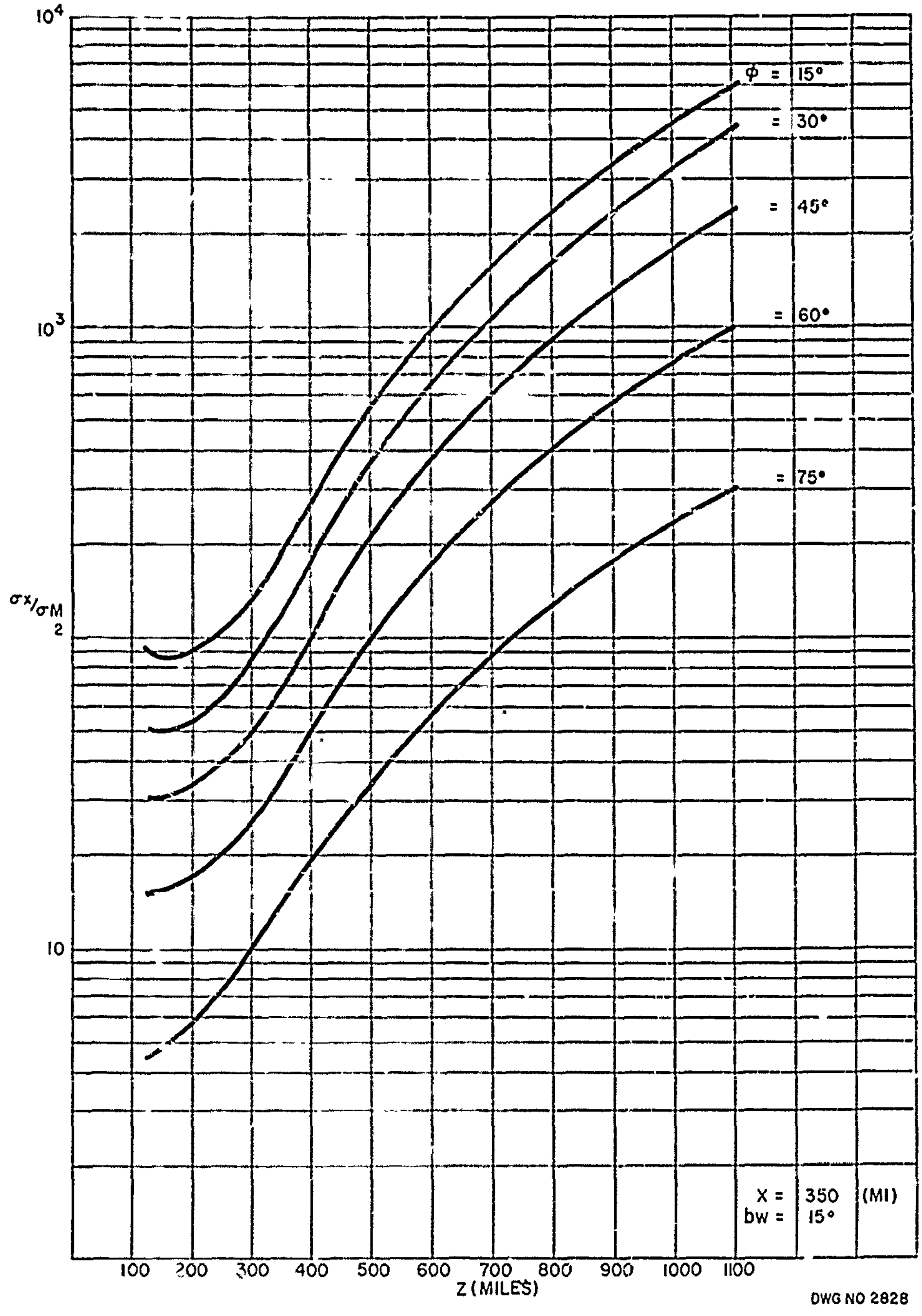
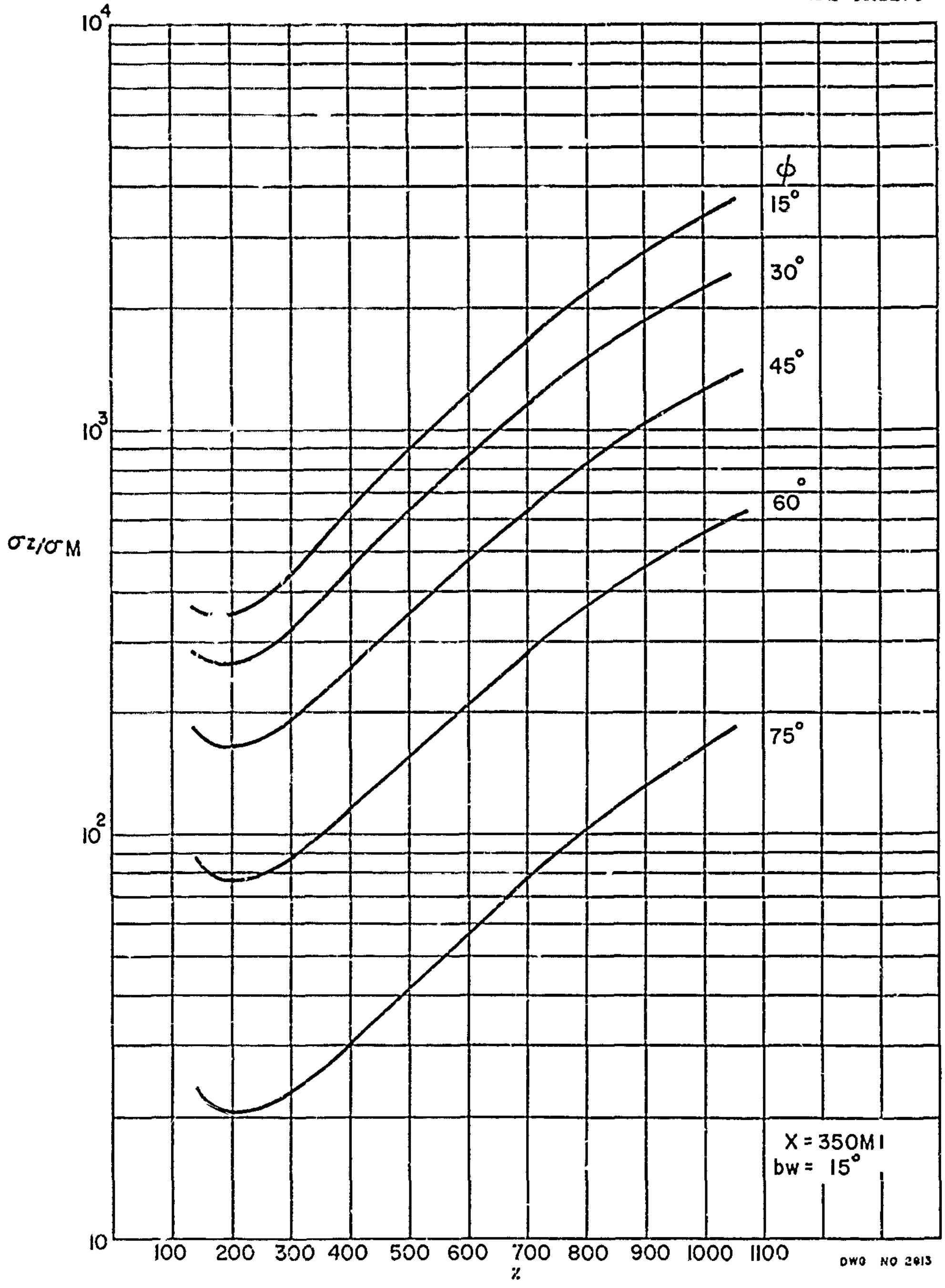
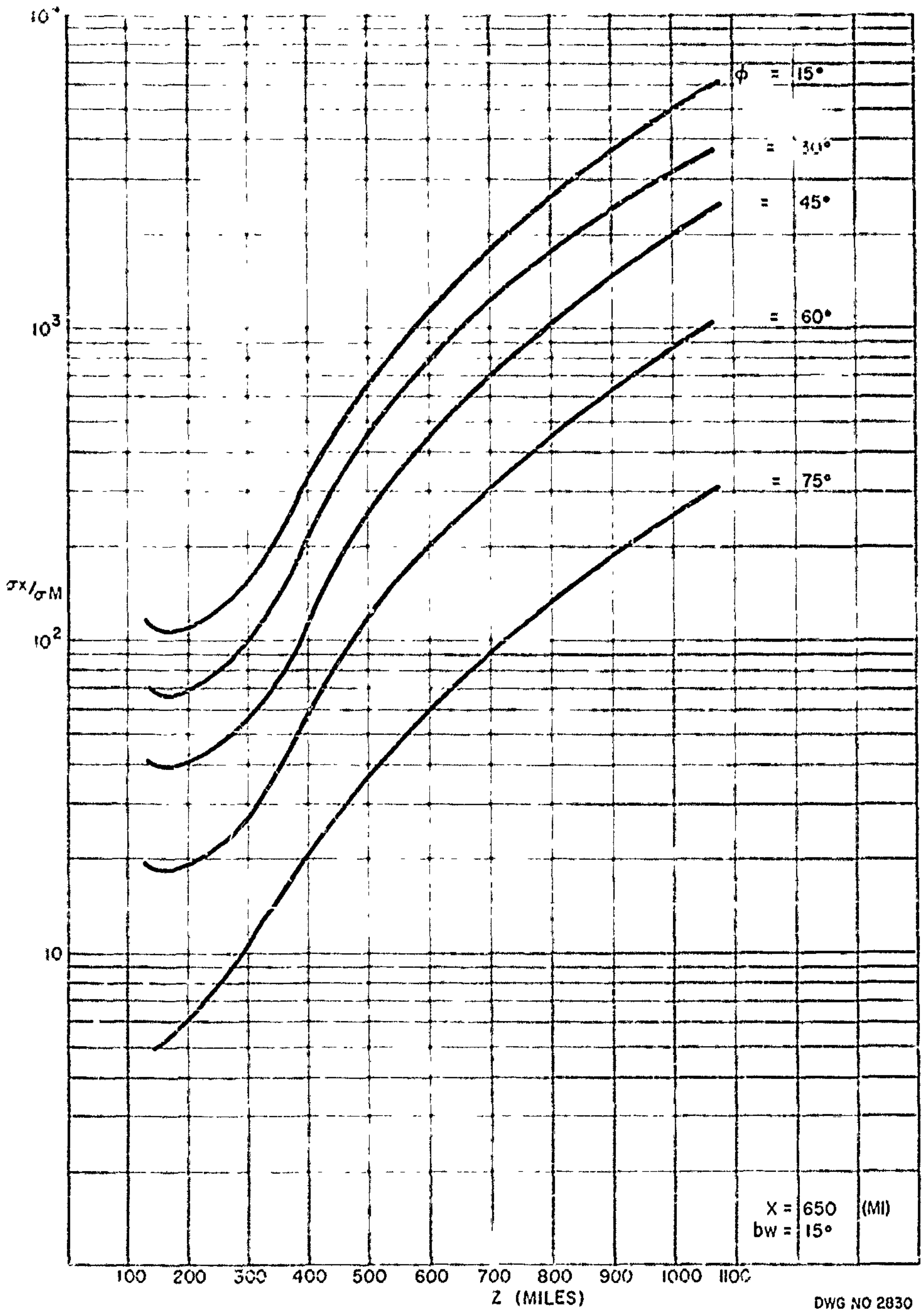

10^4
10^3
10
σx/σM
2
φ = 15°
= 30°
= 45°
= 60°
= 75°
X = 350 (MI)
bw = 15°
100 200 300 400 500 600 700 800 900 1000 1100
Z (MILES)
DWG NO 2828

Figure 19a



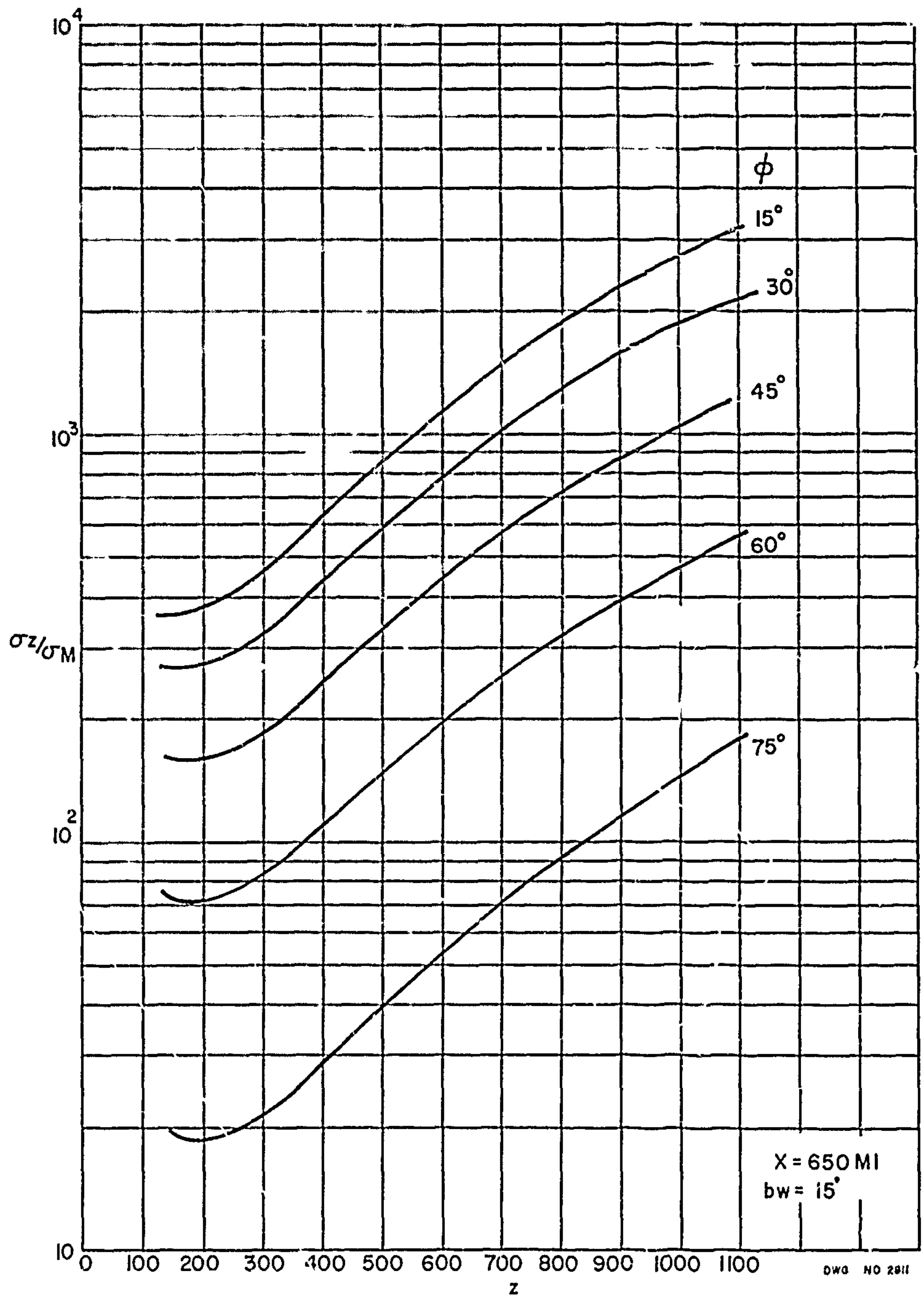
10^4
10^3
10^2
10
$\sigma z/\sigma M$
ϕ
15°
30°
45°
60°
75°
X = 350MI
bw = 15°
100 200 300 400 500 600 700 800 900 1000 1100
z
DWG NO 2913

Figure 19b



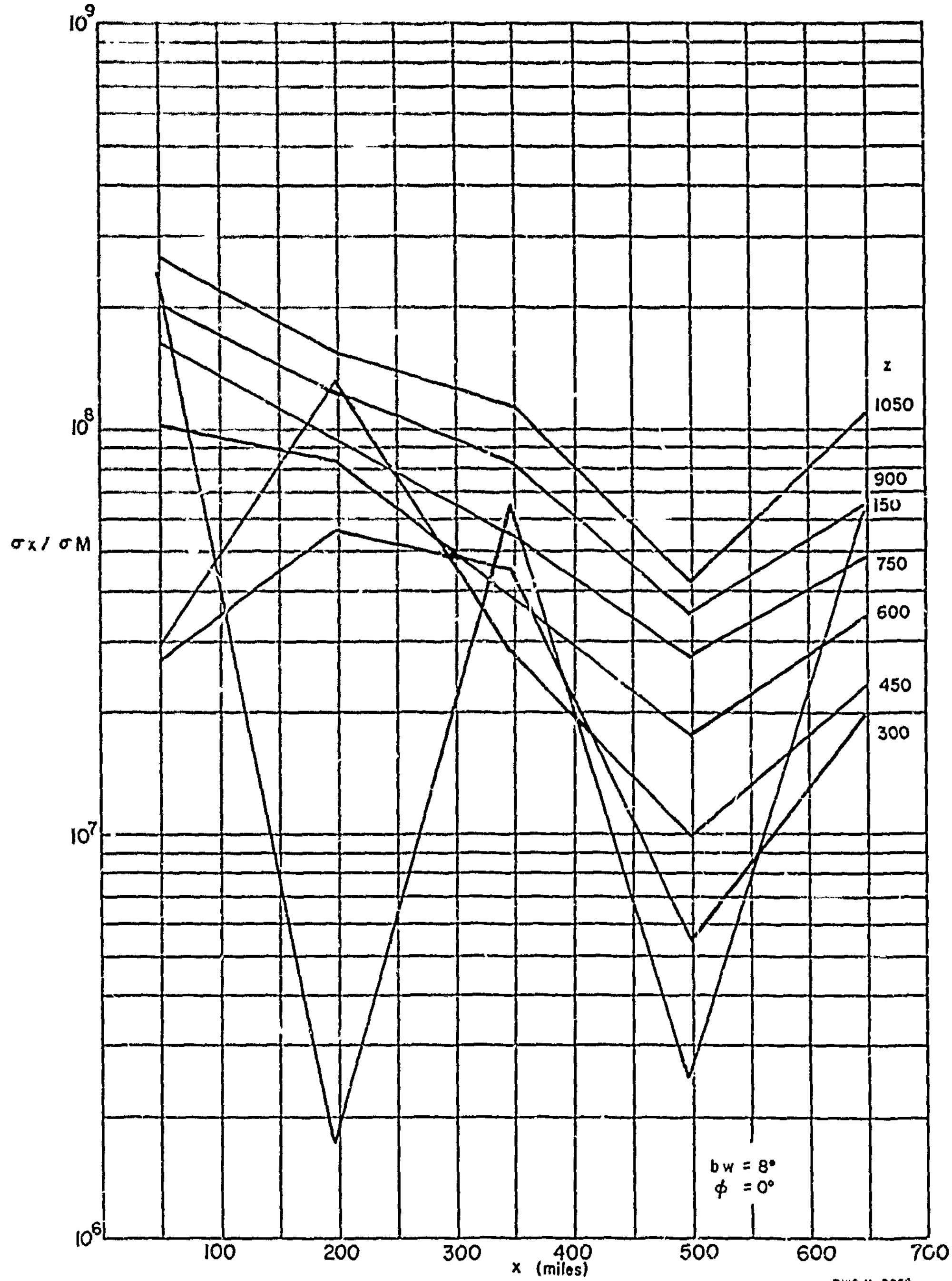
10^4
10^3
10^2
10
$\sigma x/\sigma M$
ϕ = 15°
= 30°
= 45°
= 60°
= 75°
X = 650 (MI)
bw = 15°
100 200 300 400 500 600 700 800 900 1000 1100
Z (MILES)
DWG NO 2830

Figure [illegible]



10^4
10^3
10^2
10
$\sigma z/\sigma_M$
ϕ
15°
30°
45°
60°
75°
X = 650 MI
bw = 15°
0
100
200
300
400
500
600
700
800
900
1000
1100
z
DWG NO 2911

Figure 20b



10^9
10^8
10^7
10^6
$\sigma x / \sigma M$
z
1050
900
150
750
600
450
300
bw = 8°
ϕ = 0°
100
200
300
400
500
600
700
x (miles)
DWG No 2868

Figure 21a

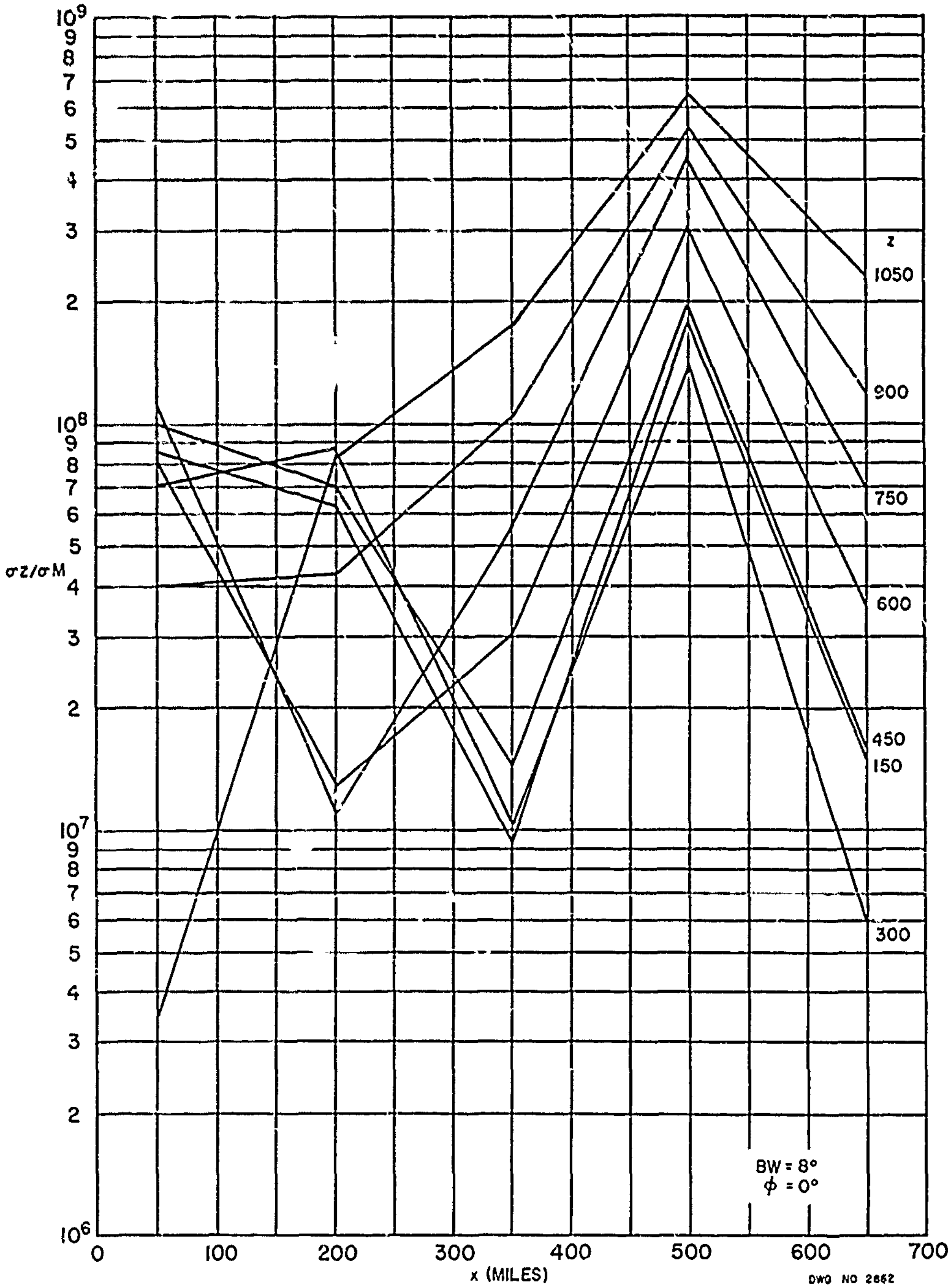
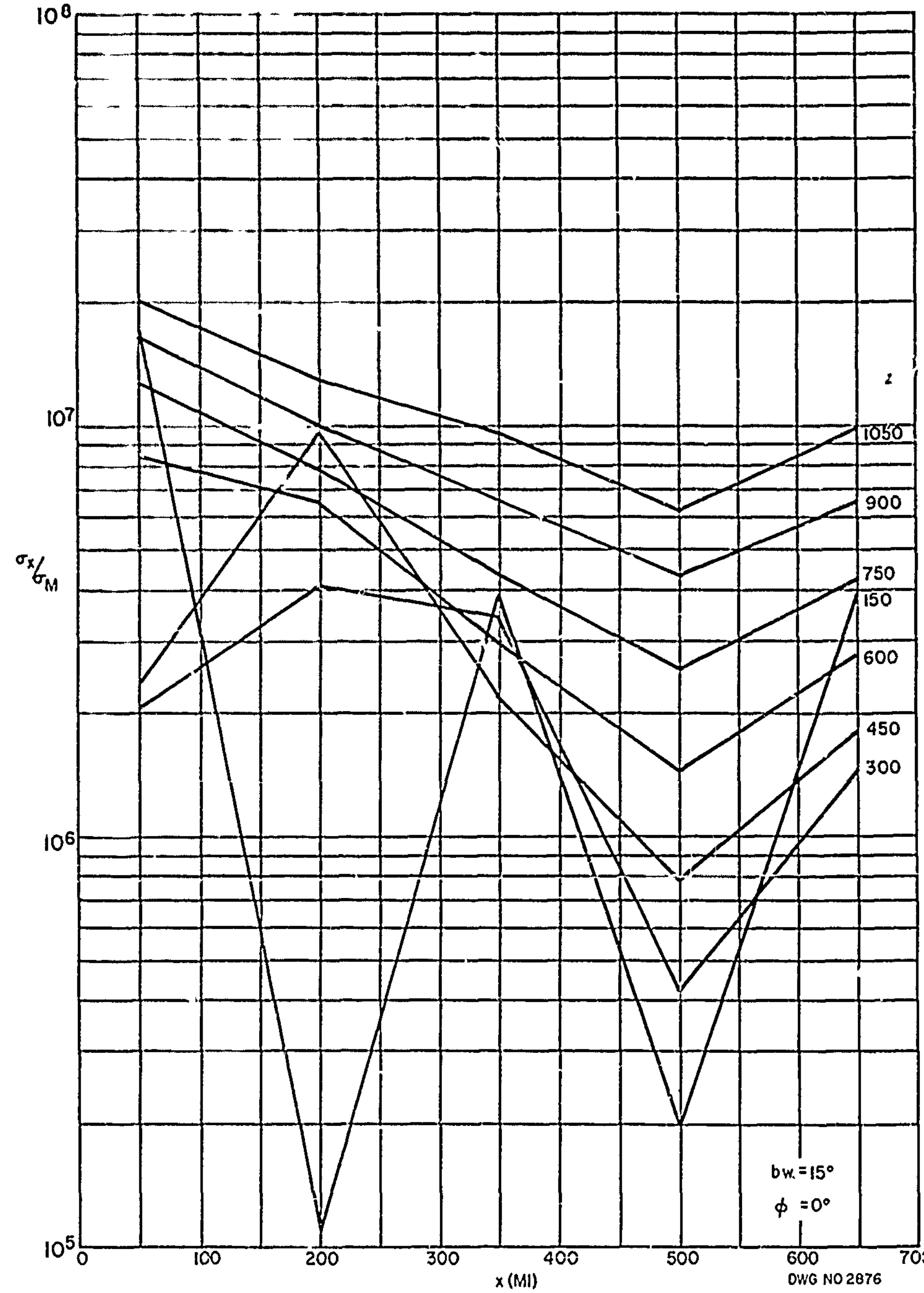


Figure 21b

C 57



10^8
10^7
10^6
10^5
σ_x/σ_M
z
1050
900
750
150
600
450
300
b.w. = 15°
ϕ = 0°
0
100
200
300
400
500
600
700
x (MI)
DWG NO 2876

Figure 22a

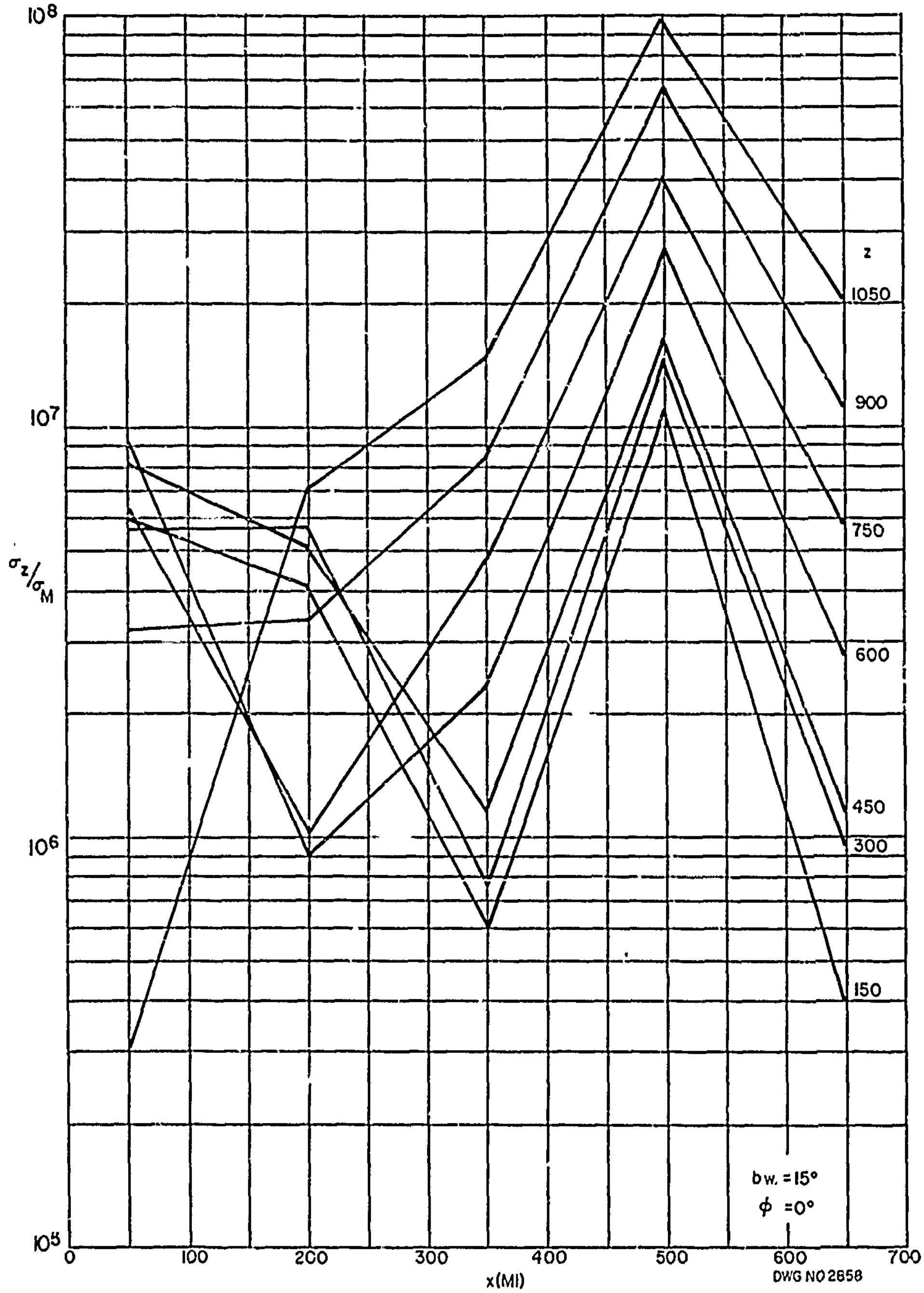

10^8
10^7
10^6
10^5
σ_z/σ_M
z
1050
900
750
600
450
300
150
b.w. =15°
ϕ =0°
0
100
200
300
400
500
600
700
x(MI)
DWG NO 2858

Figure 22b

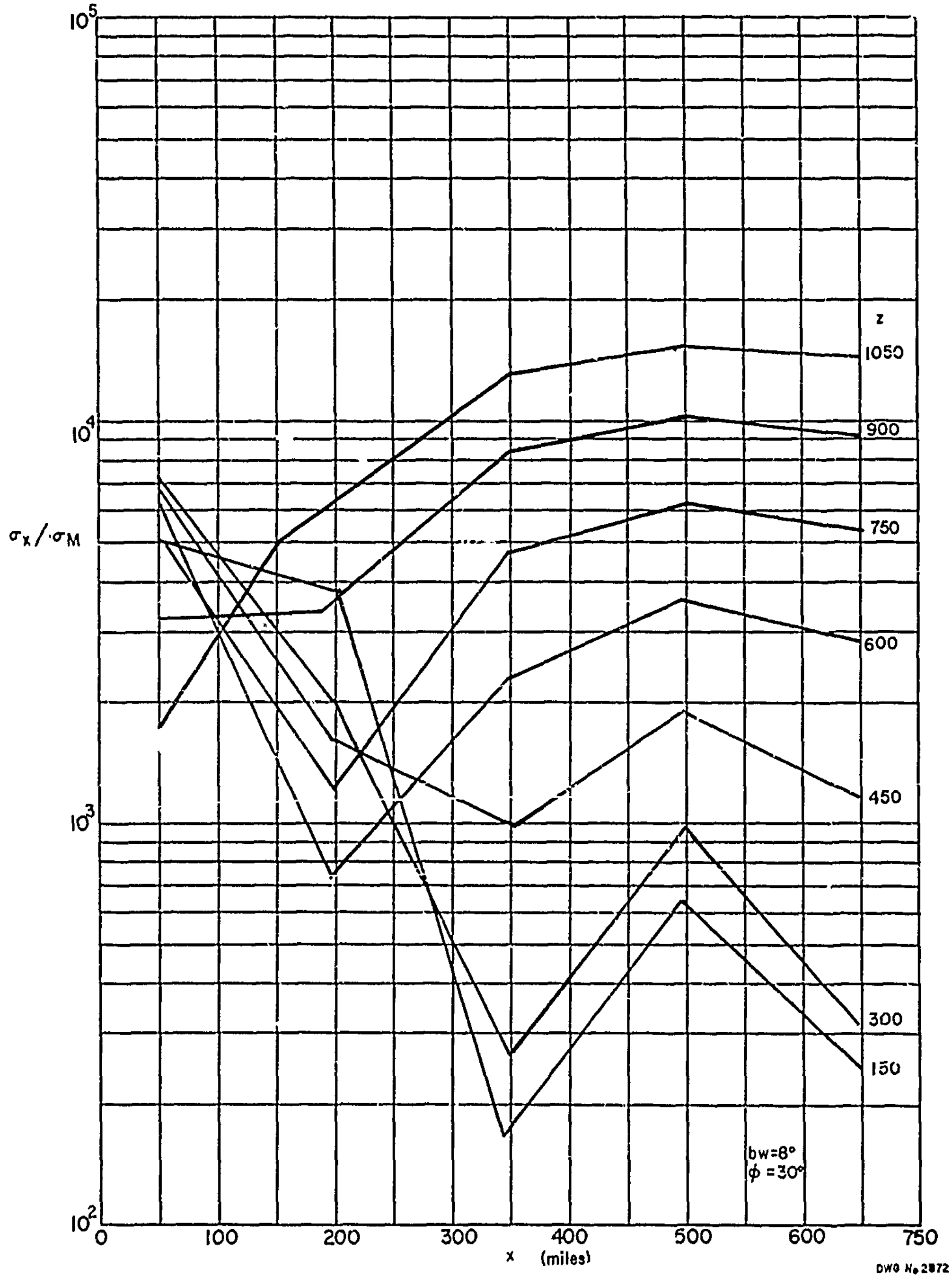
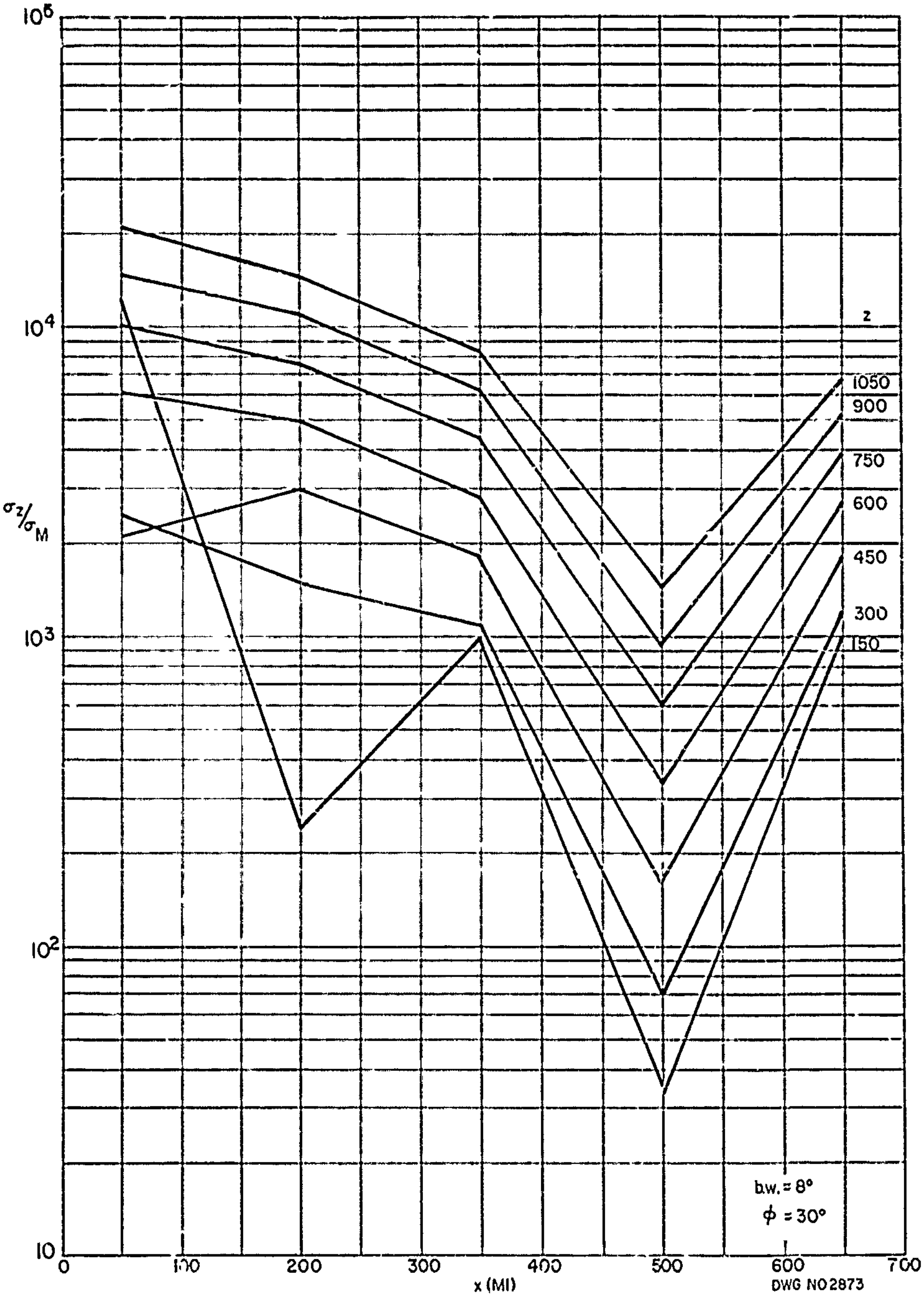
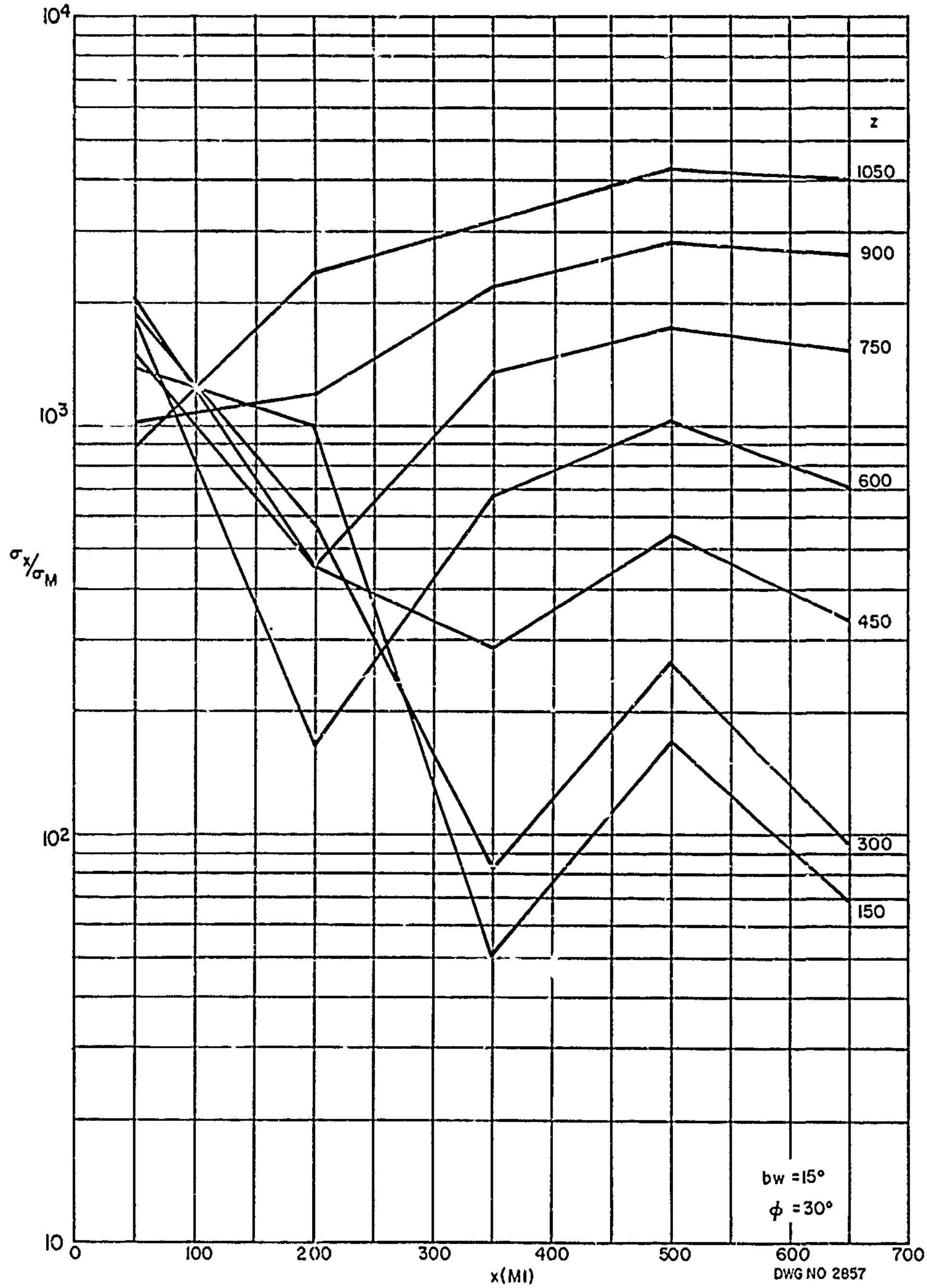

10^5
10^4
10^3
10^2
σ_x/σ_M
z
1050
900
750
600
450
300
150
bw=8°
ϕ =30°
0
100
200
300
400
500
600
750
x (miles)
DWG No. 2872

Figure 23a

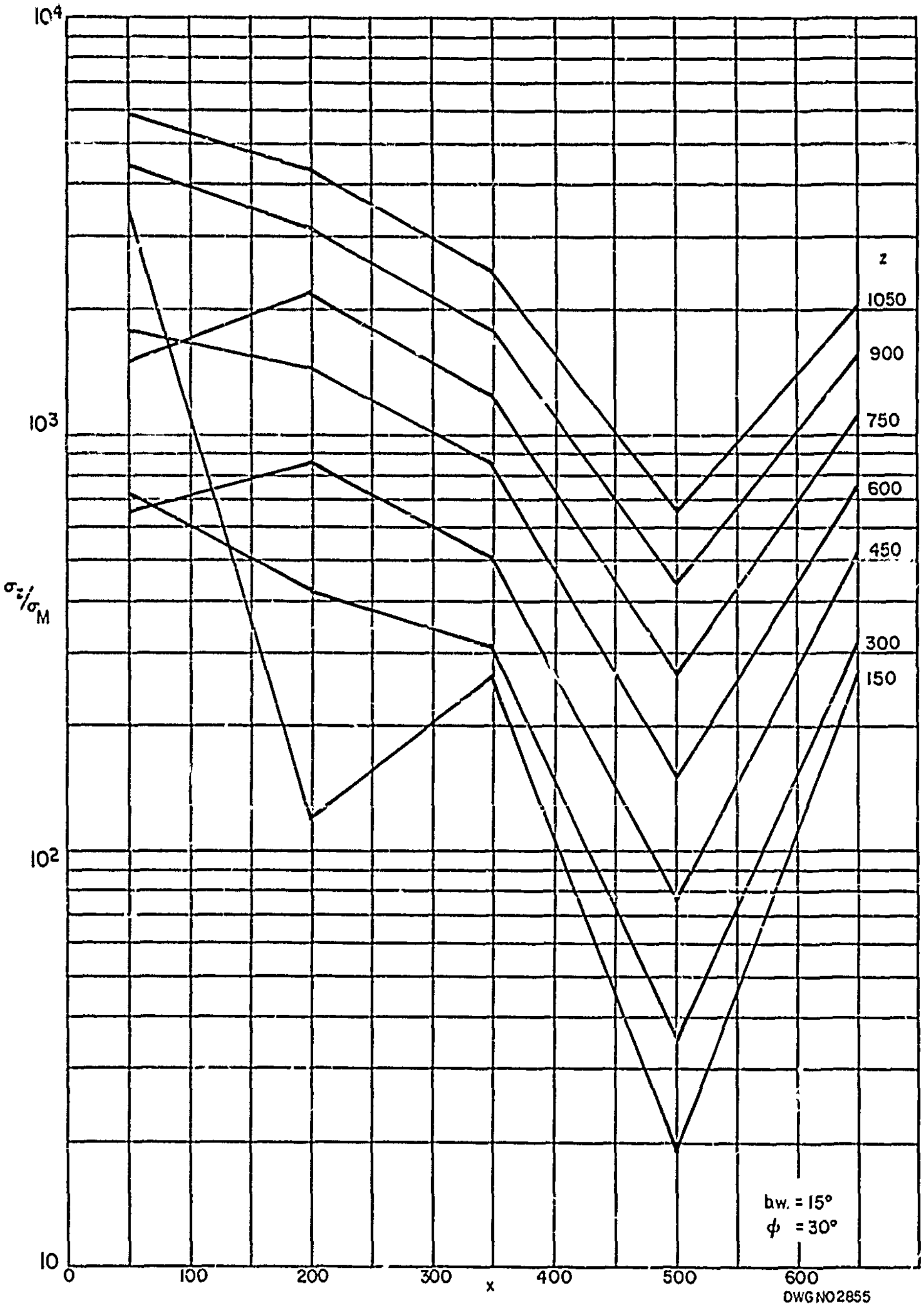


10^5
10^4
10^3
10^2
10
σ_z/σ_M
z
1050
900
750
600
450
300
150
b.w. = 8°
ϕ = 30°
0
100
200
300
400
500
600
700
x (MI)
DWG NO 2873



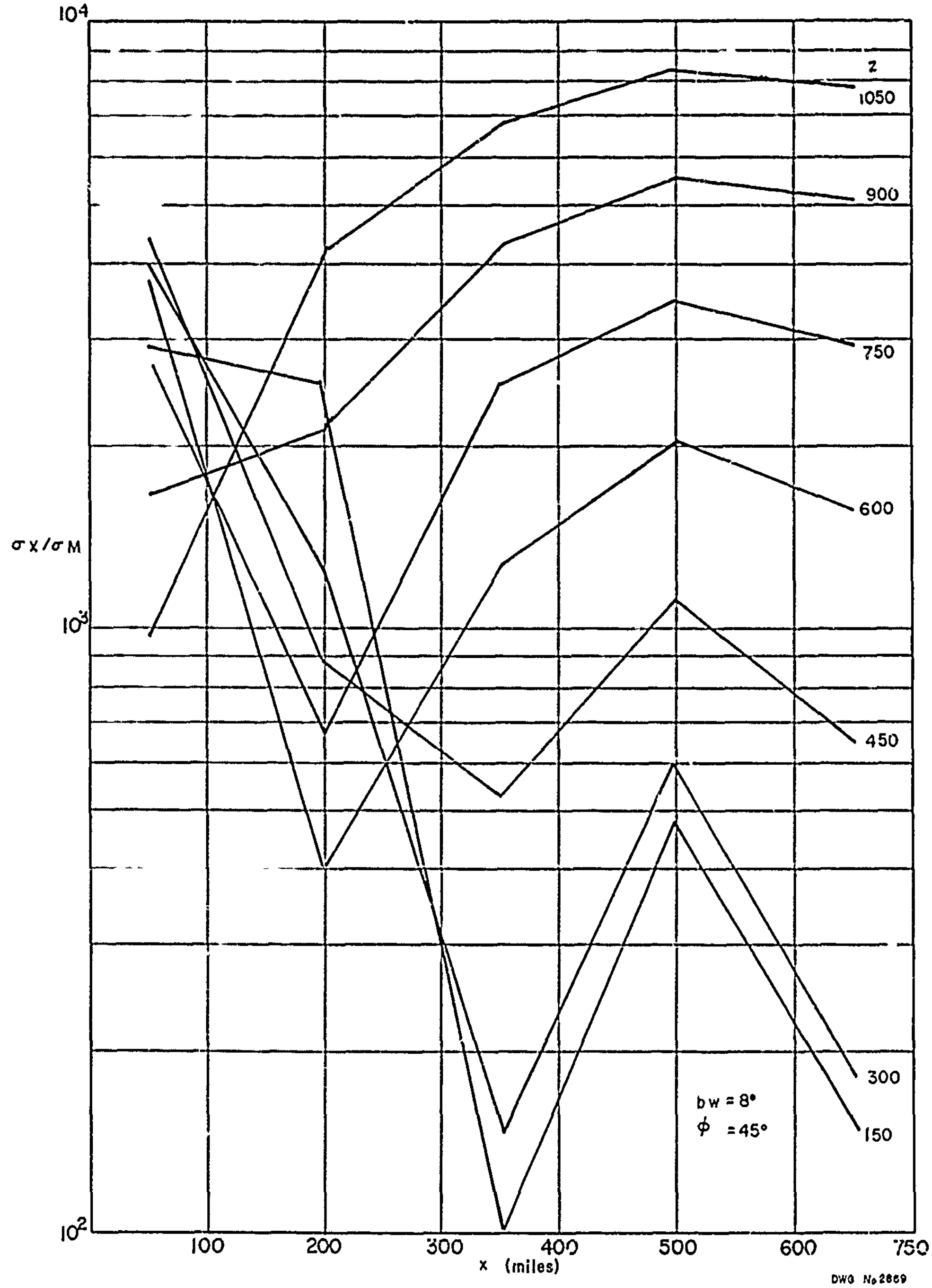
10^4
10^3
10^2
10
σ_x/σ_M
z
1050
900
750
600
450
300
150
bw =15°
ϕ =30°
0
100
200
300
400
500
600
700
x(MI)
DWG NO 2857

Figure 24a



10⁴
10³
10²
10
σ_z/σ_M
z
1050
900
750
600
450
300
150
b.w. = 15°
ϕ = 30°
0
100
200
300
x
400
500
600
DWG NO2855

Figure 24b



10⁴
z
1050
900
750
600
σX/σM
10³
450
300
bw = 8°
φ = 45°
150
10²
100
200
300
400
500
600
750
x (miles)
DWG No 2869

Figure 25a

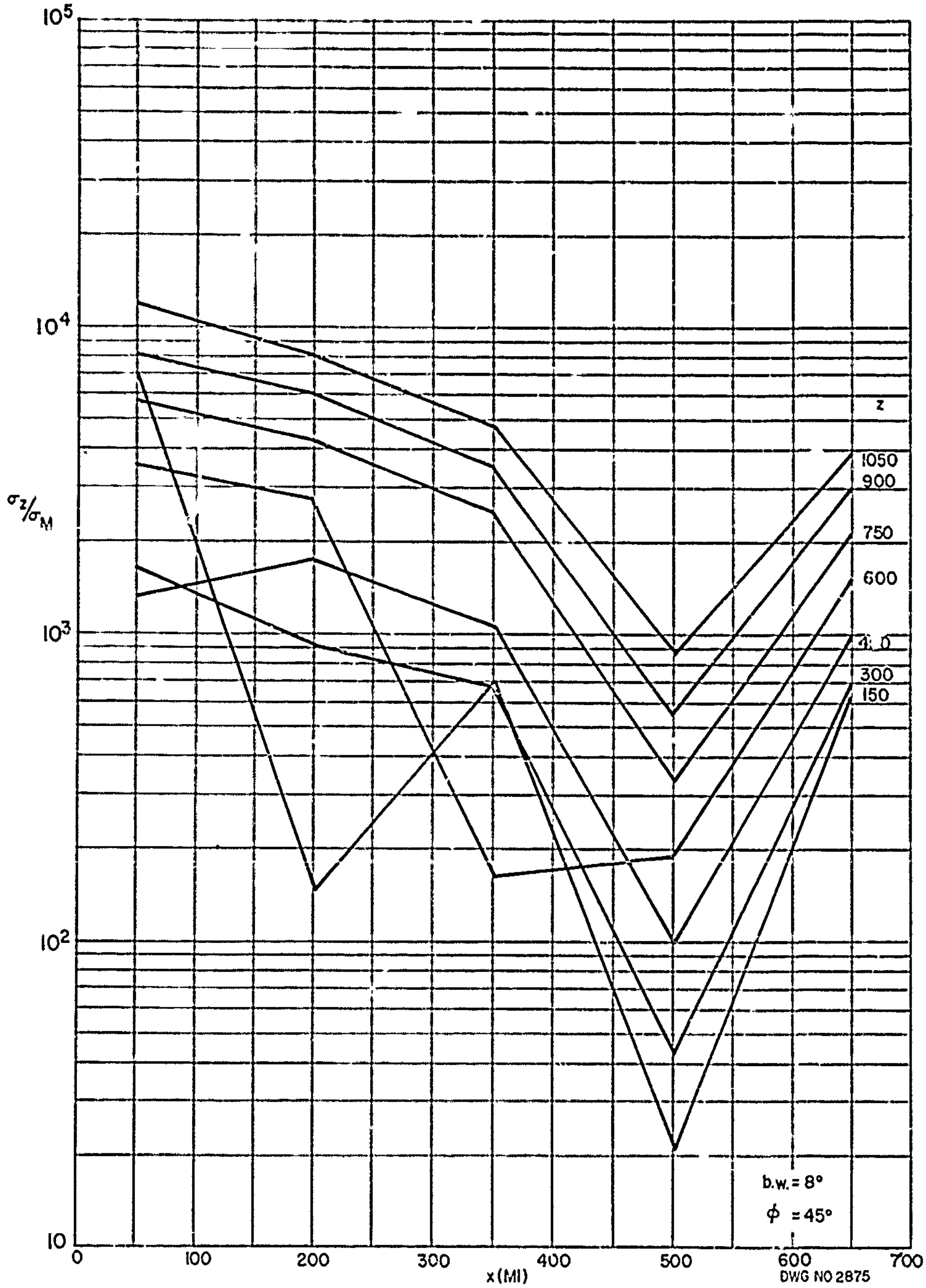
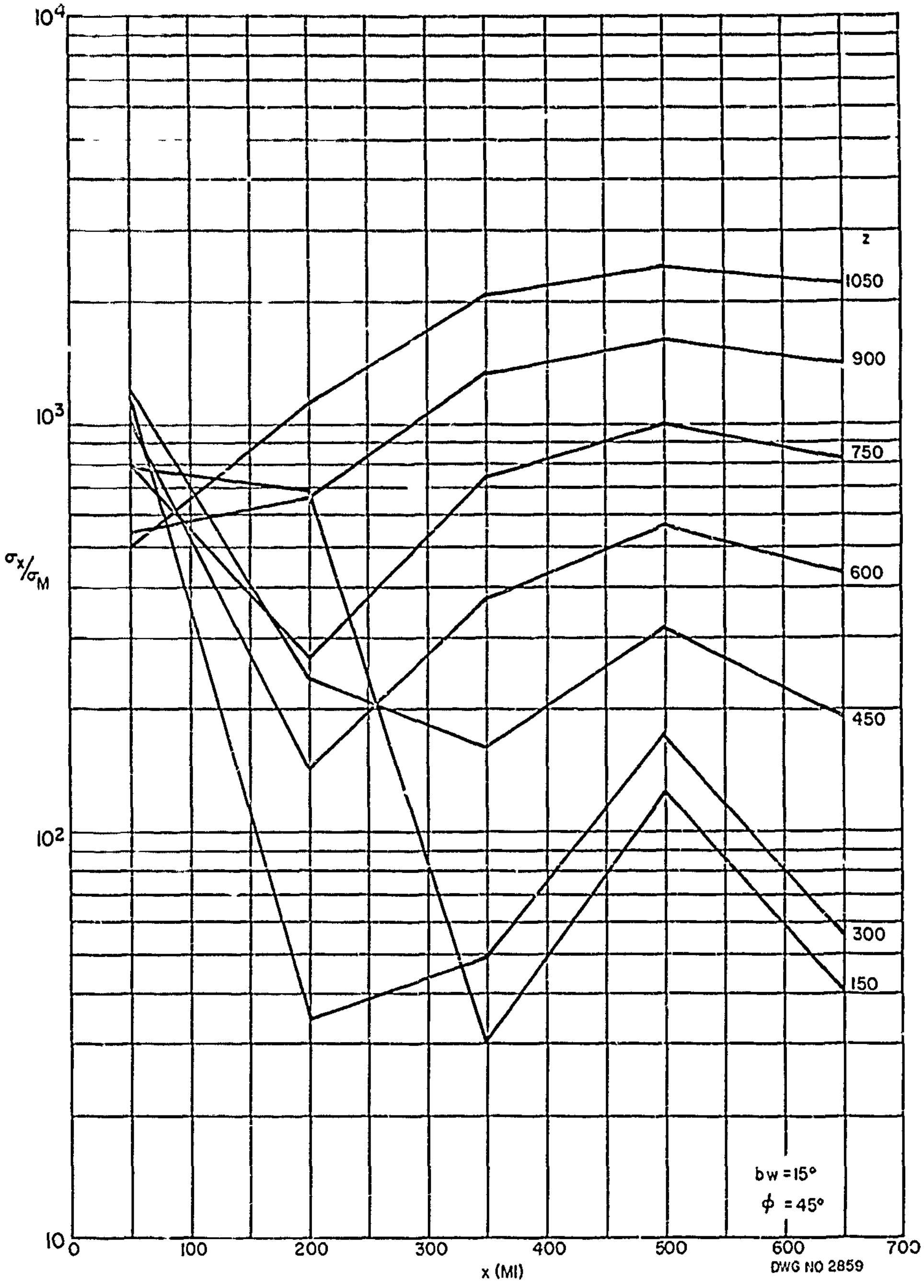

10^5
10^4
10^3
10^2
10
σz/σM
0
100
200
300
400
500
600
700
x (MI)
DWG NO 2875
z
1050
900
750
600
450
300
150
b.w. = 8°
φ = 45°

Figure 25b



10^4
10^3
10^2
10
σ_x/σ_M
z
1050
900
750
600
450
300
150
bw = 15°
ϕ = 45°
0
100
200
300
400
500
600
700
x (MI)
DWG NO 2859

Figure 26a

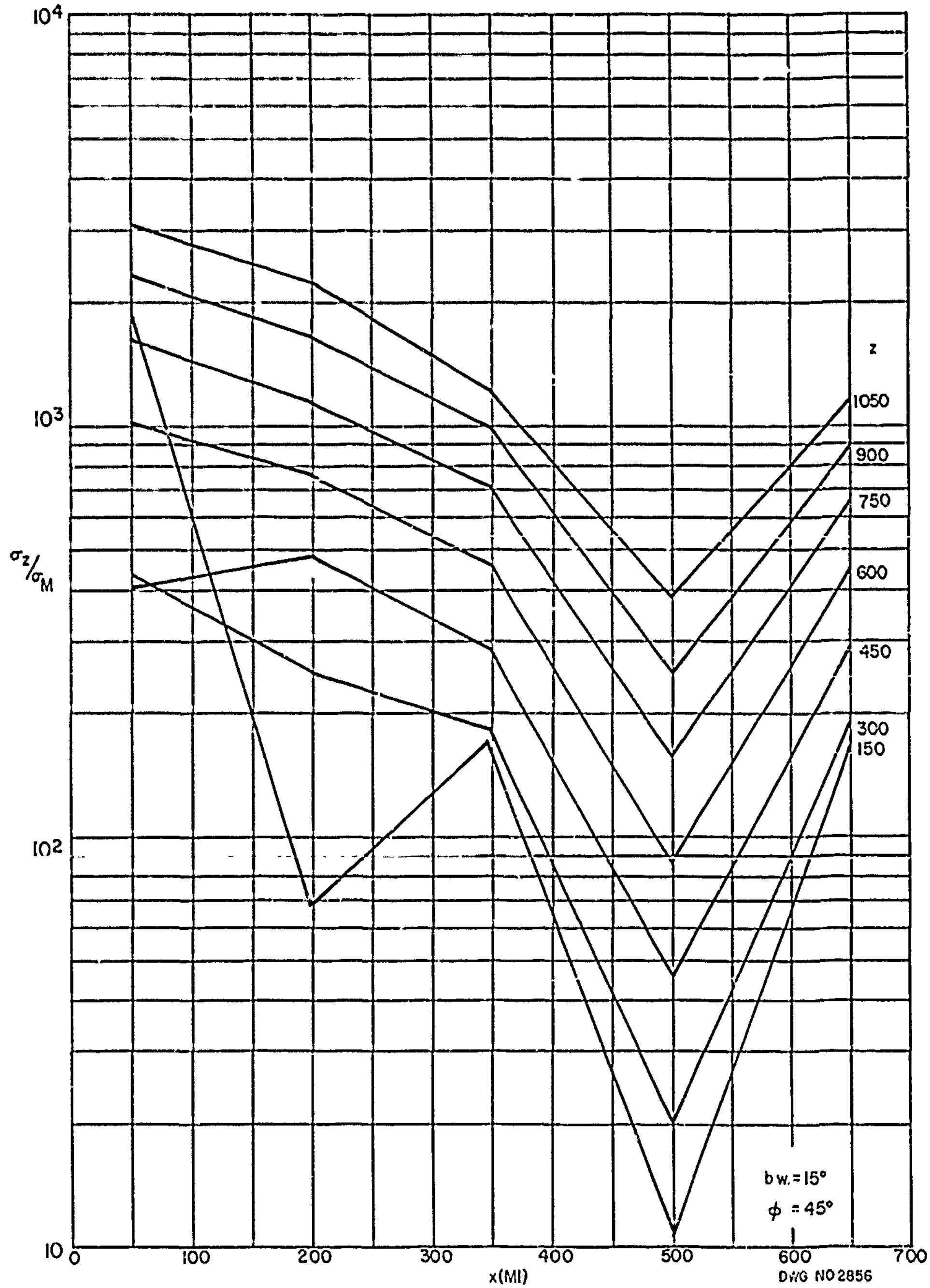

10^4
10^3
10^2
10
σ_z/σ_M
z
1050
900
750
600
450
300
150
b w. = 15°
ϕ = 45°
0
100
200
300
400
500
600
700
x(MI)
DWG NO 2856

Figure 26b

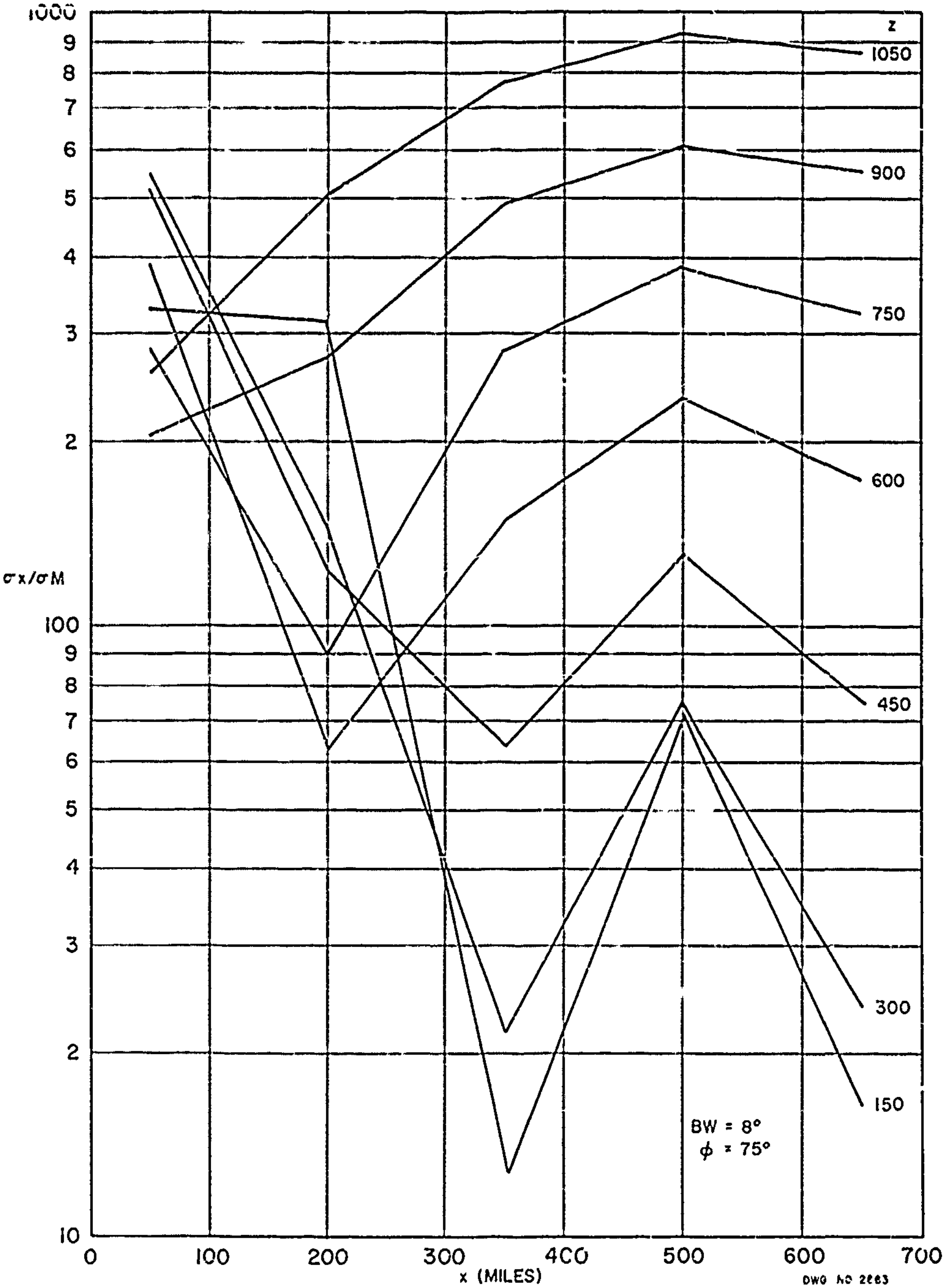
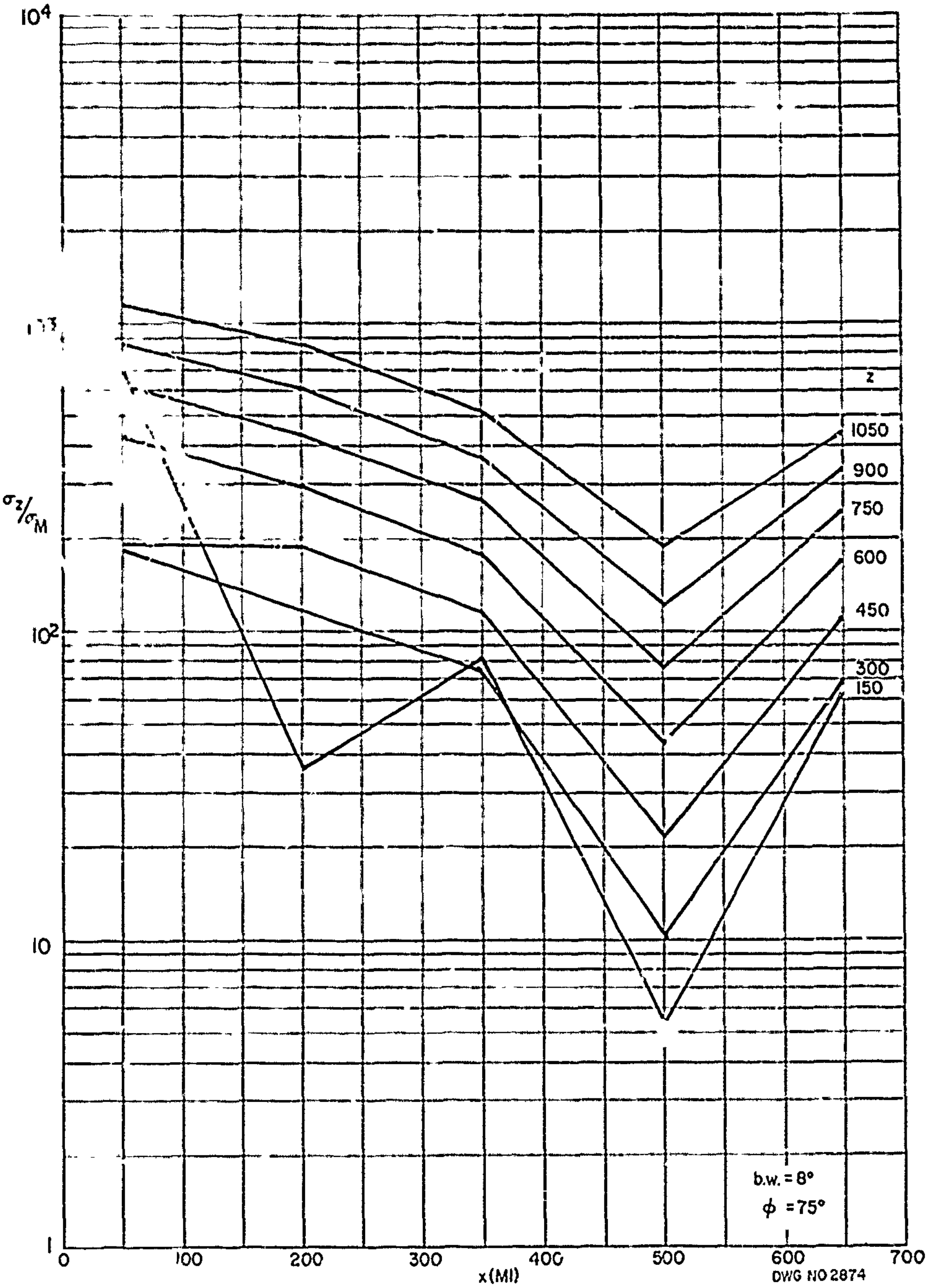
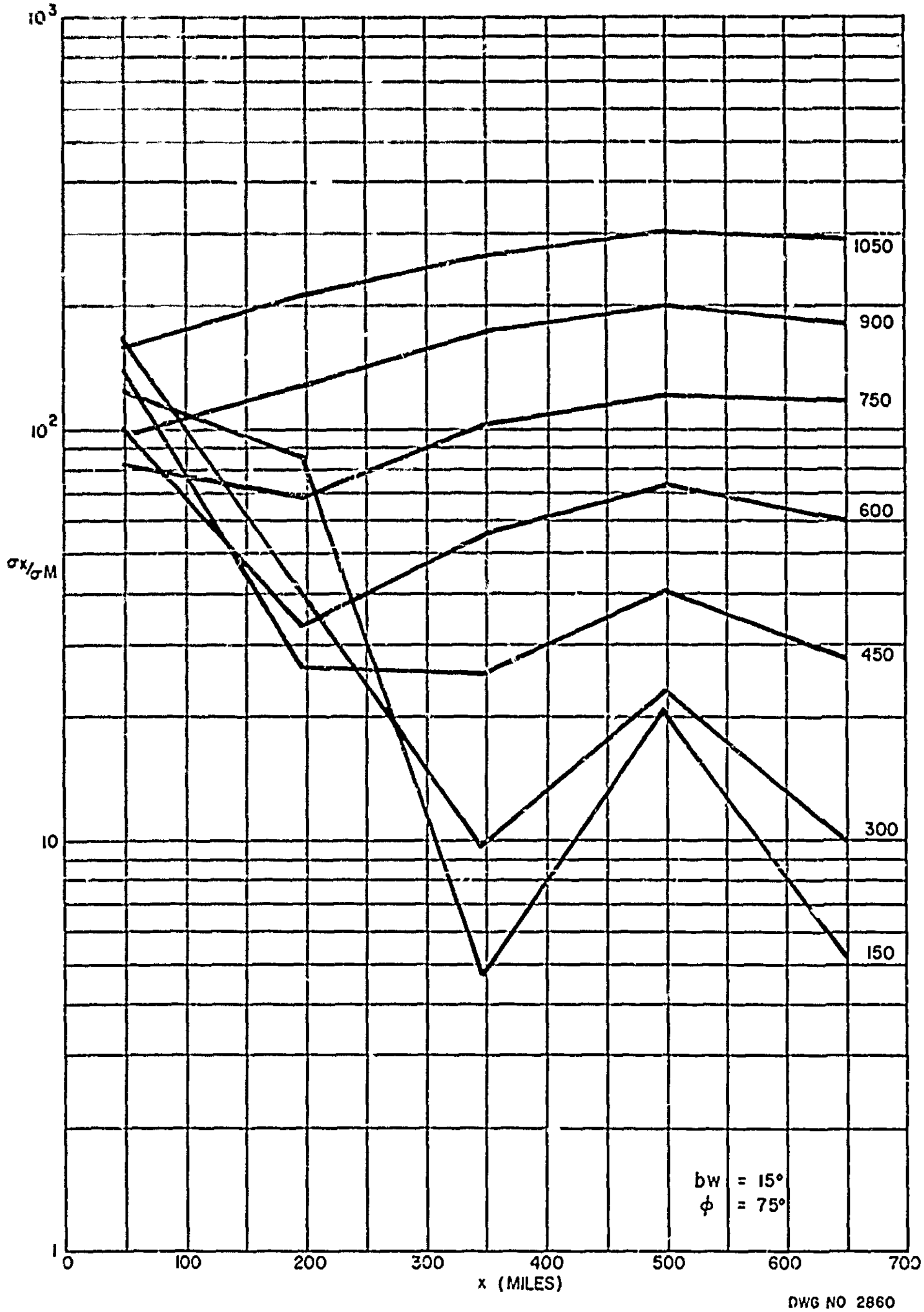

z
1050
900
750
600
450
300
150
BW = 8°
φ = 75°
σx/σM
1000
9
8
7
6
5
4
3
2
100
9
8
7
6
5
4
3
2
10
0
100
200
300
400
500
600
700
x (MILES)
DWG NO 2883

Figure 27a



10^4
10^3
10^2
10
1
σz/σM
z
1050
900
750
600
450
300
150
b.w. = 8°
φ = 75°
0
100
200
300
400
500
600
700
x(MI)
DWG NO 2874

Figure 27b



10^3
10^2
10
1
$\sigma x/\sigma M$
1050
900
750
600
450
300
150
bw = 15°
ϕ = 75°
0
100
200
300
400
500
600
700
x (MILES)
DWG NO 2860

Figure 28a

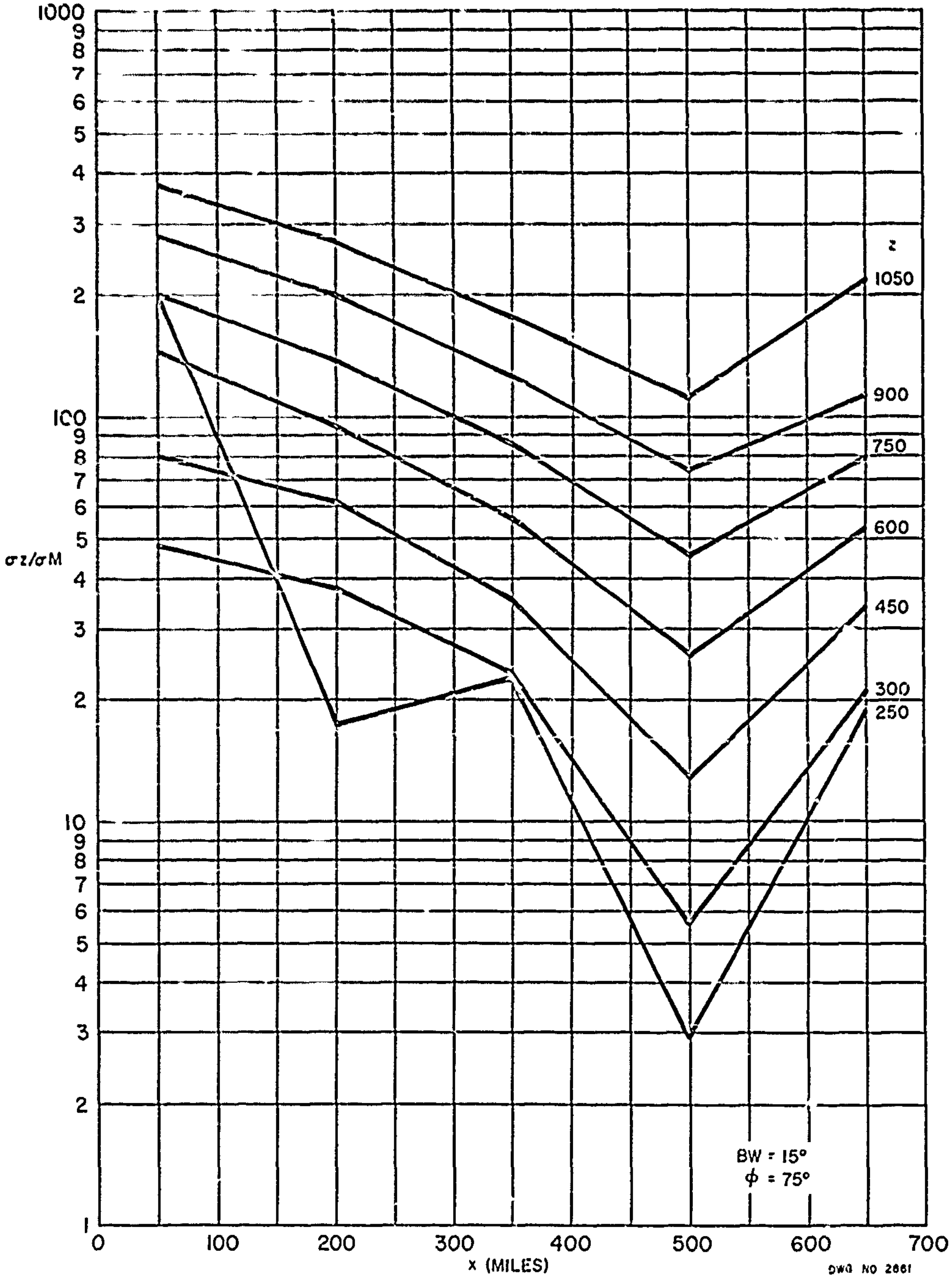
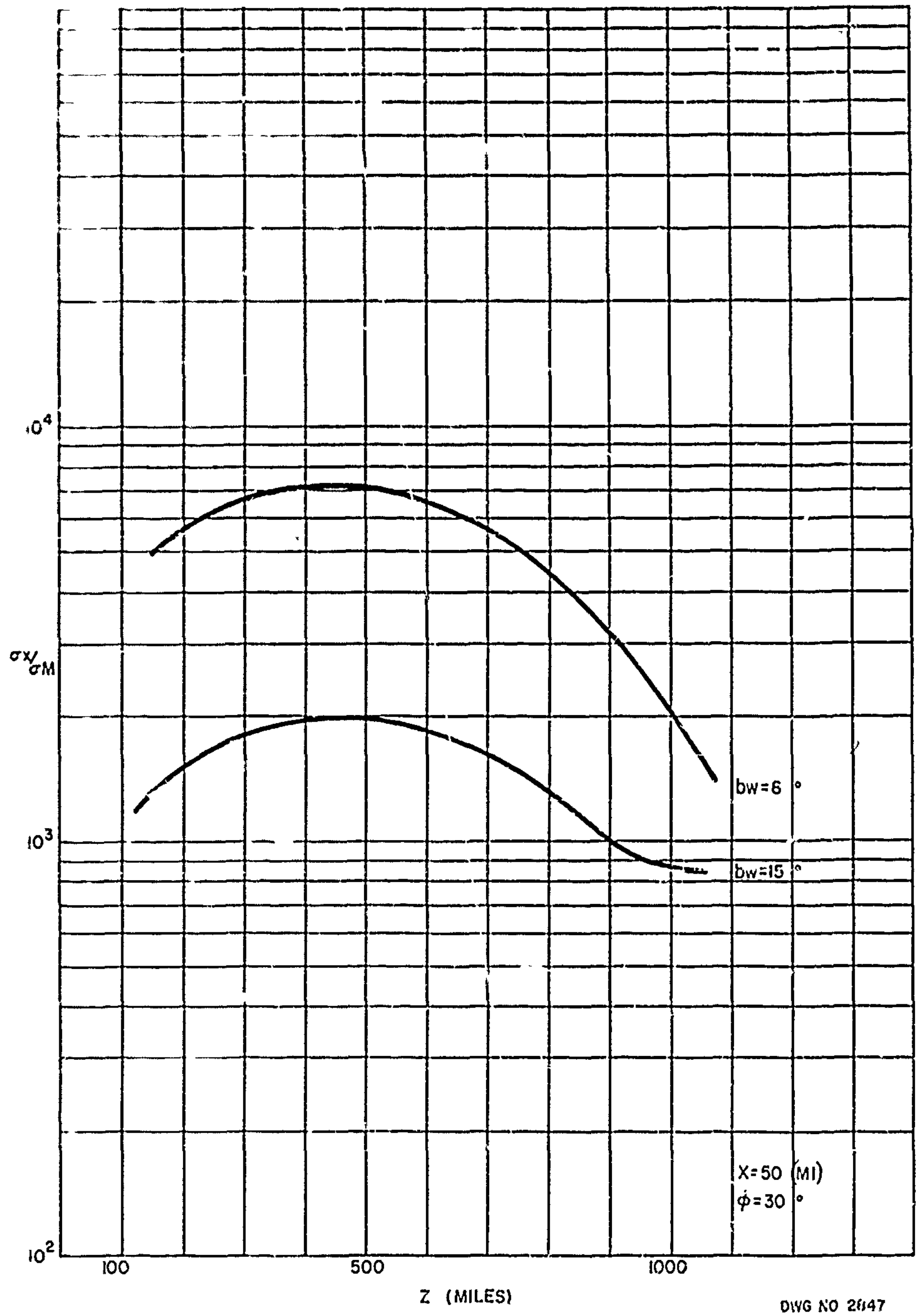
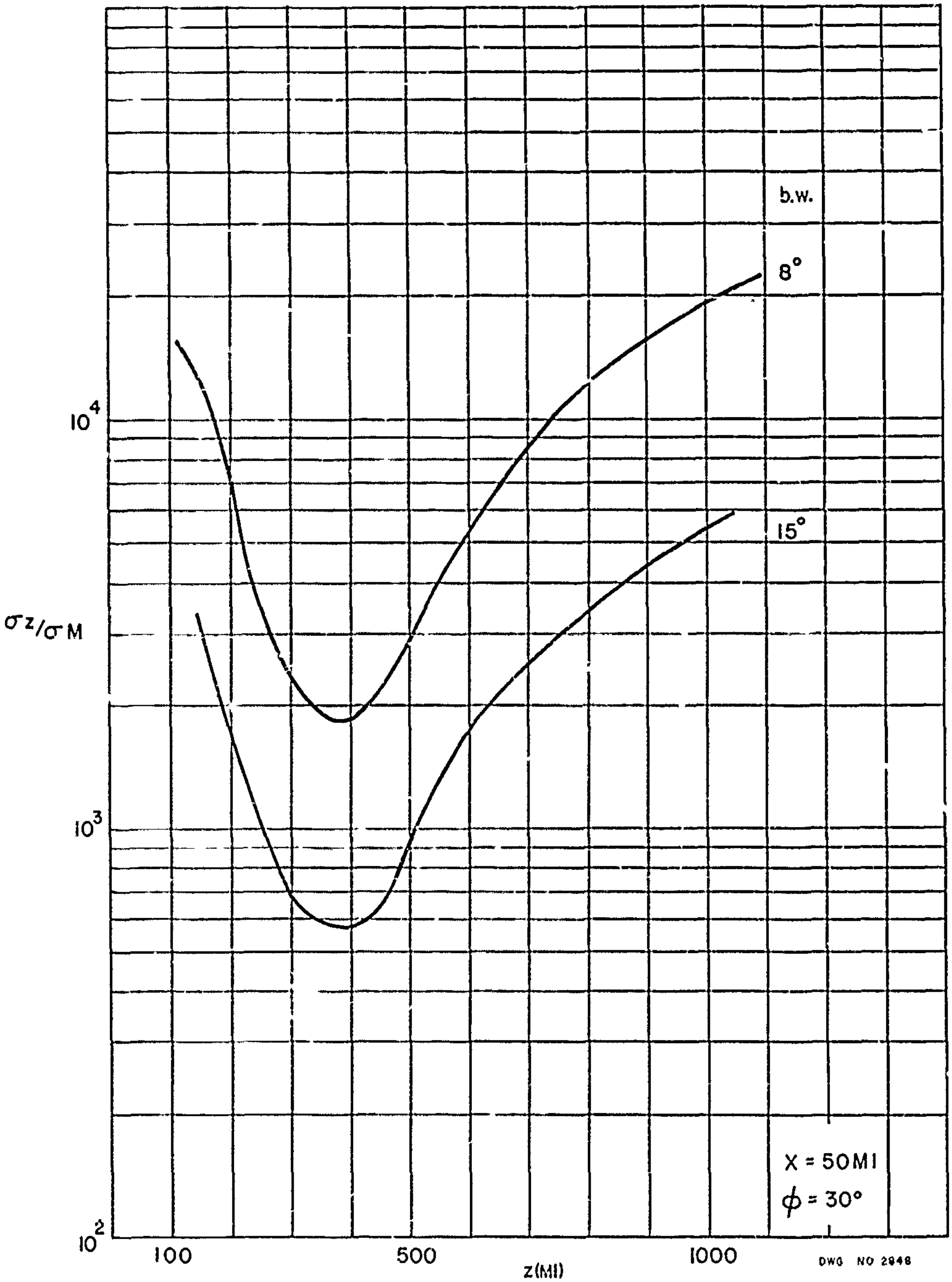

1000
9
8
7
6
5
4
3
2
100
10
1
σz/σM
z
1050
900
750
600
450
300
250
BW = 15°
φ = 75°
0
100
200
300
400
500
600
700
x (MILES)
DWG NO 2661

Figure 28b



DWG NO 2047

Figure 29a



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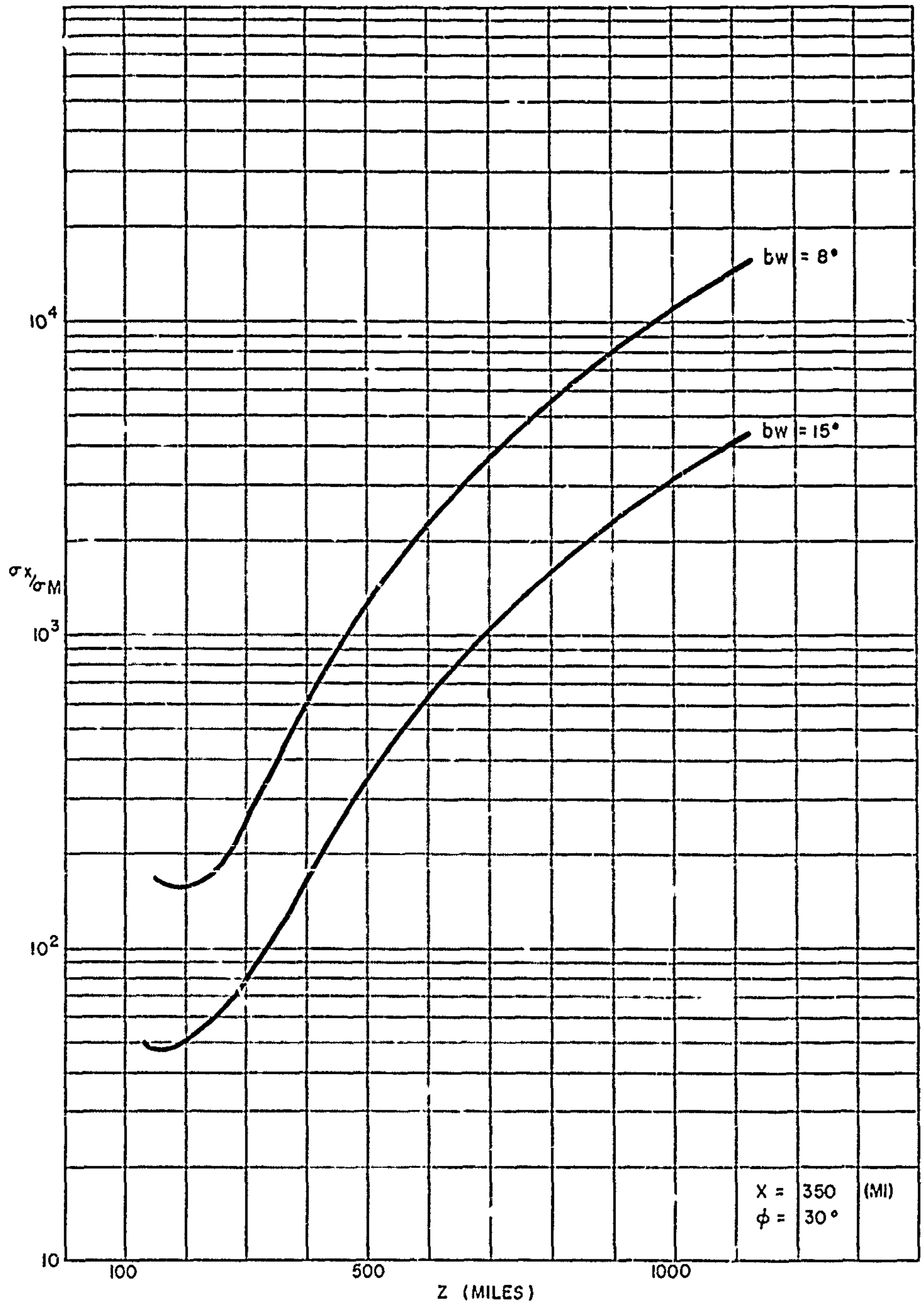
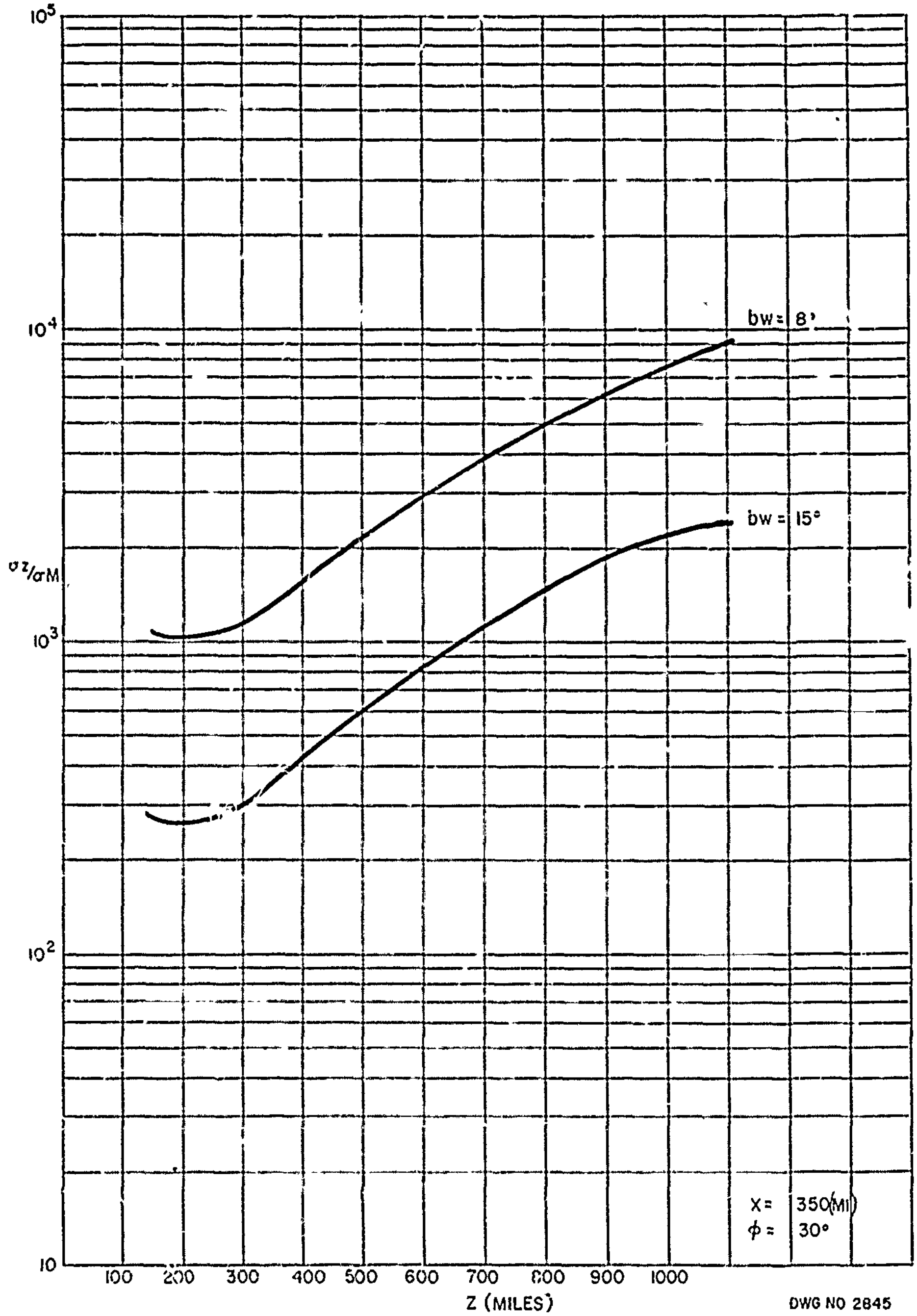

bw = 8°
bw = 15°
10^4
$\sigma x/\sigma M$
10^3
10^2
10
X = 350 (MI)
ϕ = 30°
100
500
1000
Z (MILES)

Figure 30a



10^5
10^4
10^3
10^2
10
$\sigma z/\sigma M$
bw = 8°
bw = 15°
X = 350(MI)
ϕ = 30°
100 200 300 400 500 600 700 800 900 1000
Z (MILES)
DWG NO 2845

Figure 30b

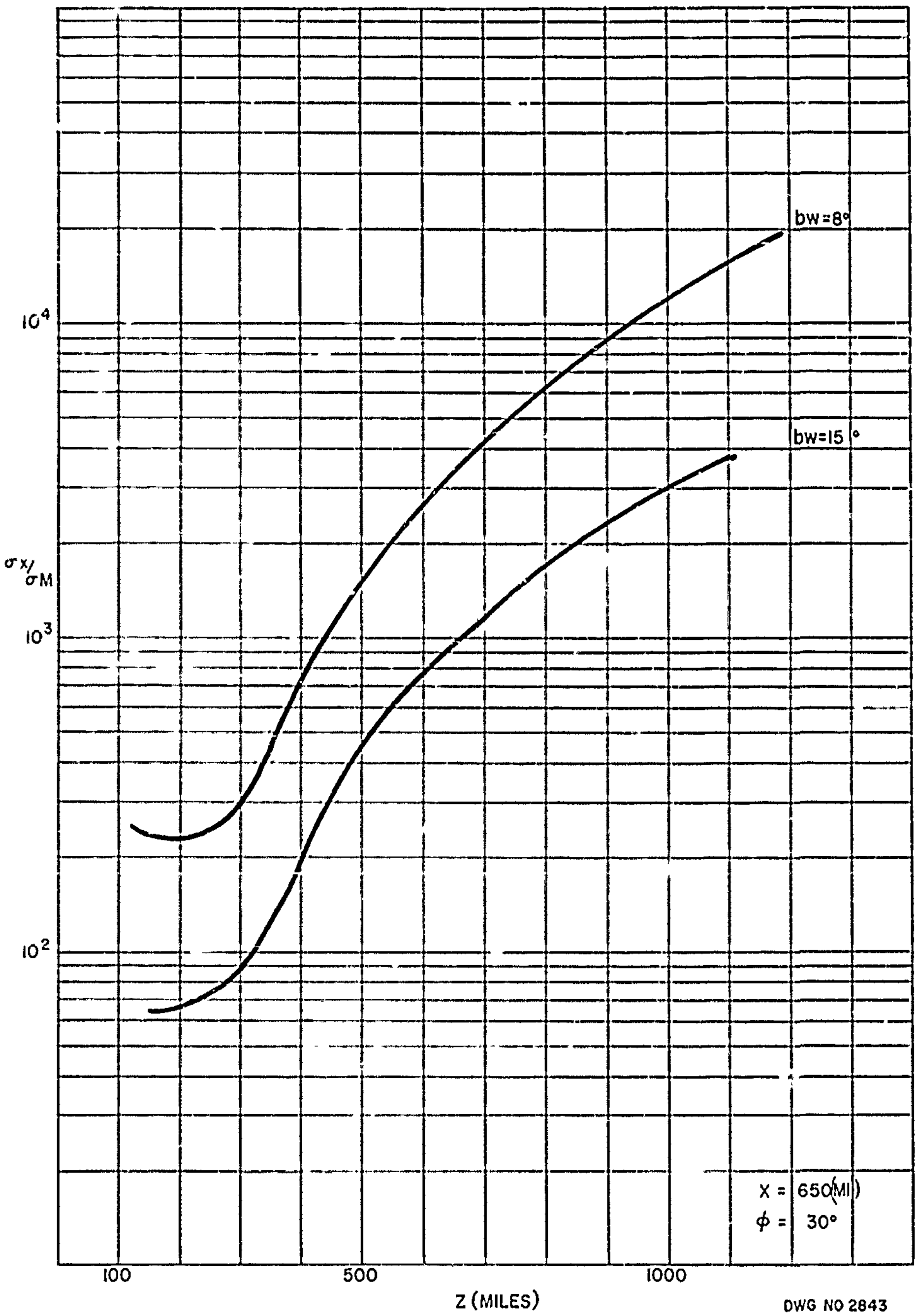

bw=8°
bw=15°
10^4
$\sigma x / \sigma M$
10^3
10^2
X = 650(MI)
φ = 30°
100
500
1000
Z (MILES)
DWG NO 2843

Figure 31a

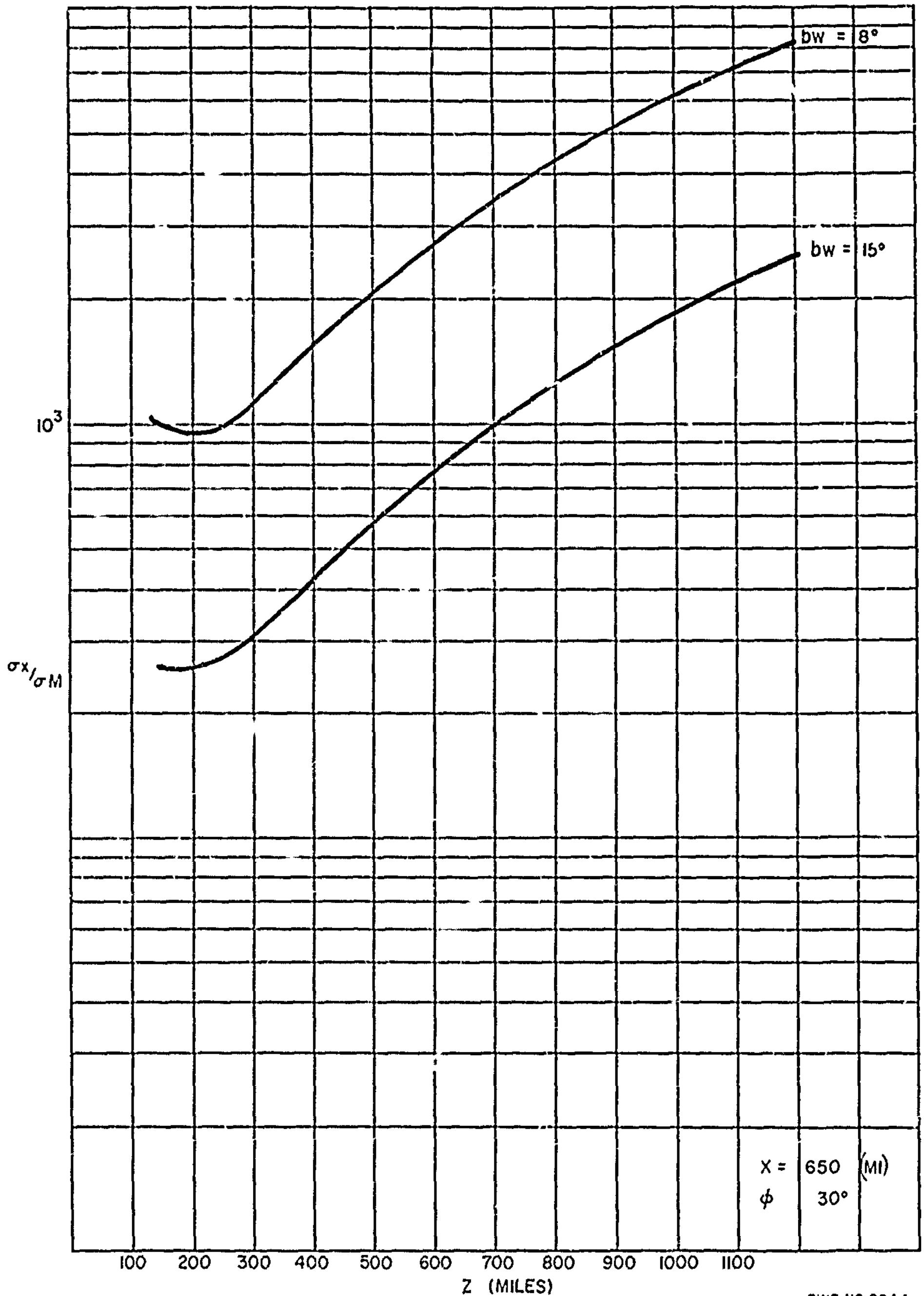
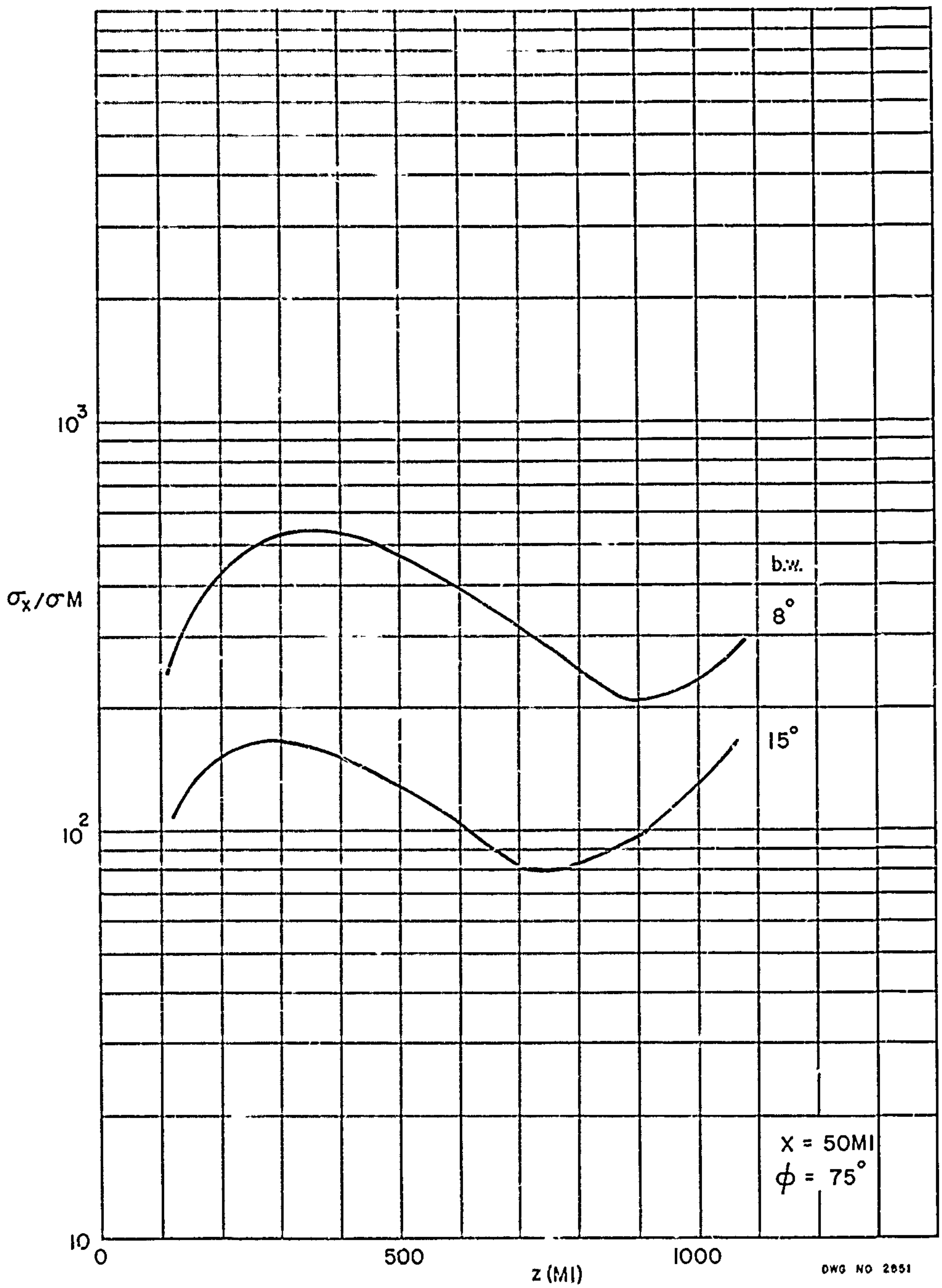
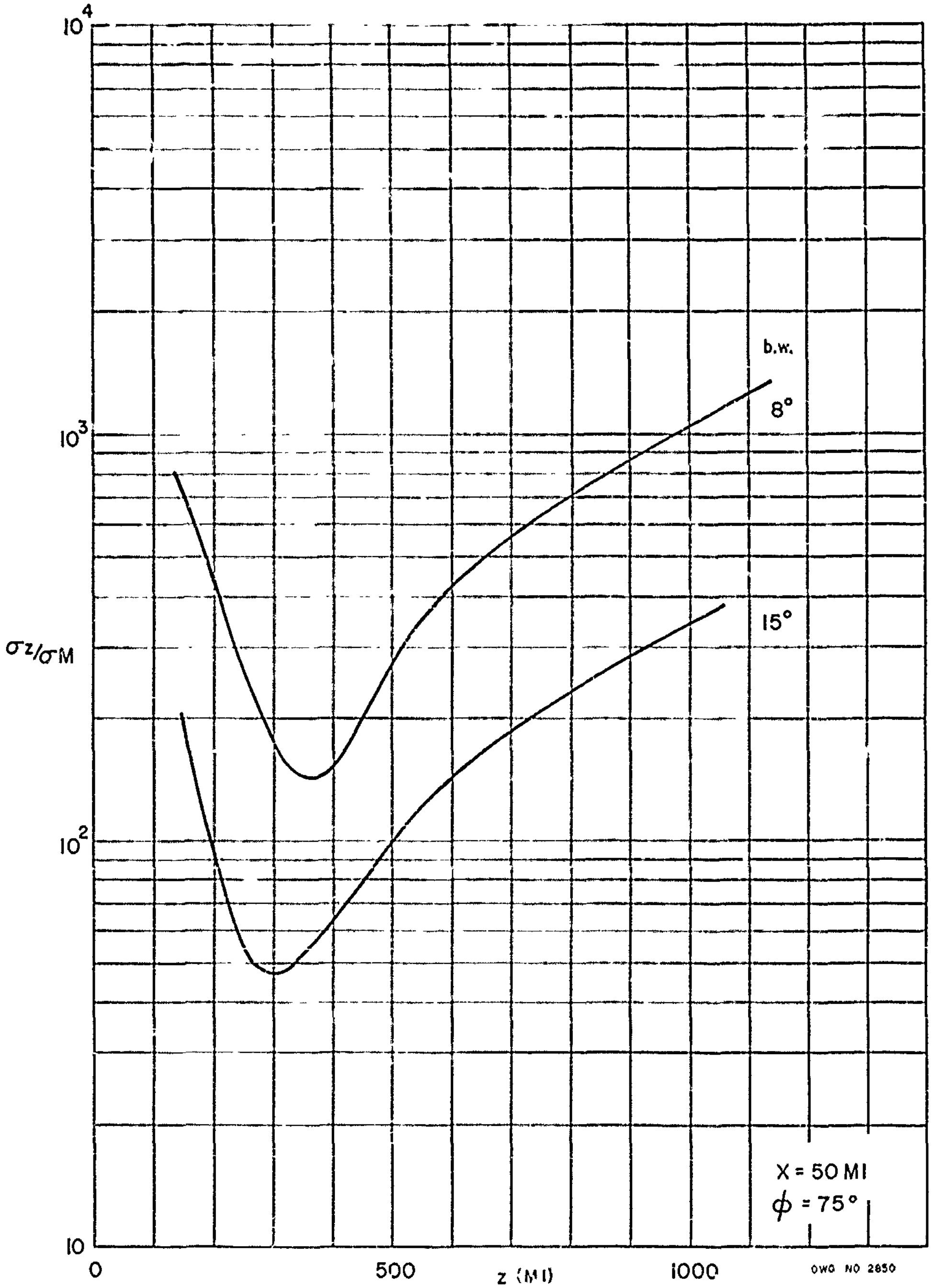

bw = 8°
bw = 15°
10^3
$\sigma x / \sigma M$
X = 650 (MI)
ϕ 30°
100 200 300 400 500 600 700 800 900 1000 1100
Z (MILES)
DWG NO 2844

Figure 31b



10^3
10^2
10
$\sigma_x/\sigma M$
b.w.
8°
15°
X = 50MI
ϕ = 75°
0
500
1000
z (MI)
DWG NO 2651


10^4
10^3
10^2
10
σz/σM
b.w.
8°
15°
X = 50 MI
φ = 75°
0
500
1000
z (MI)
DWG NO 2850

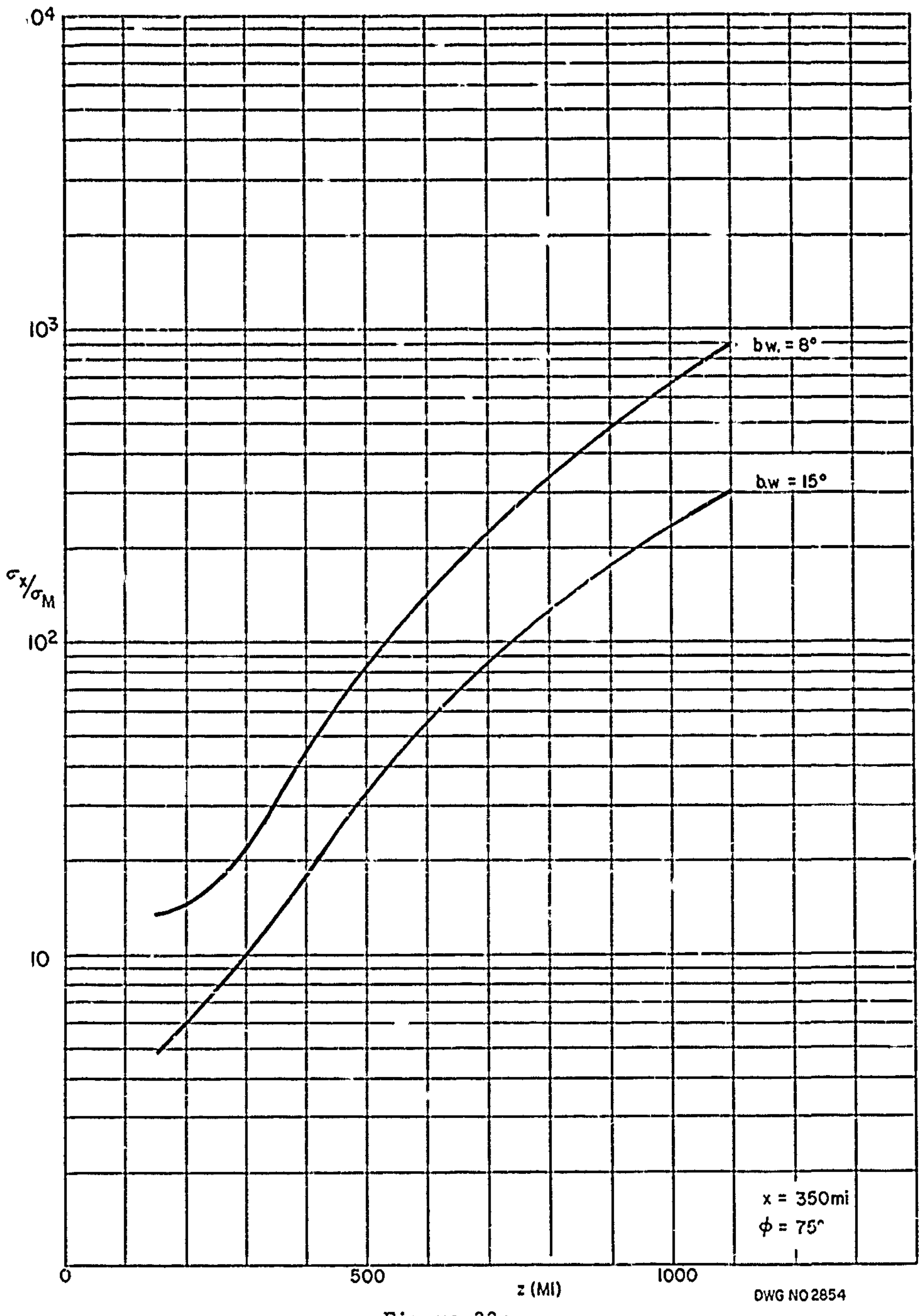
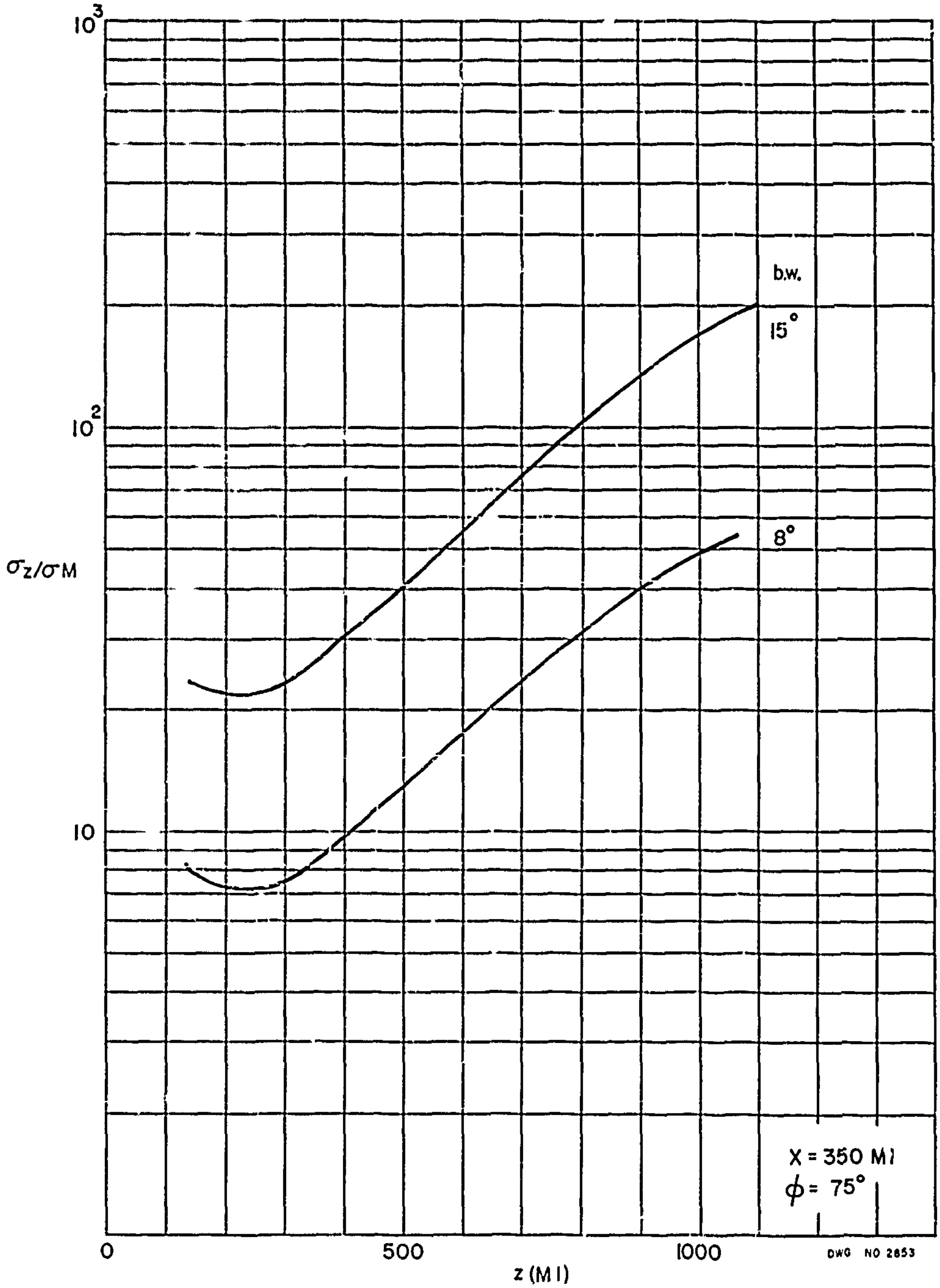
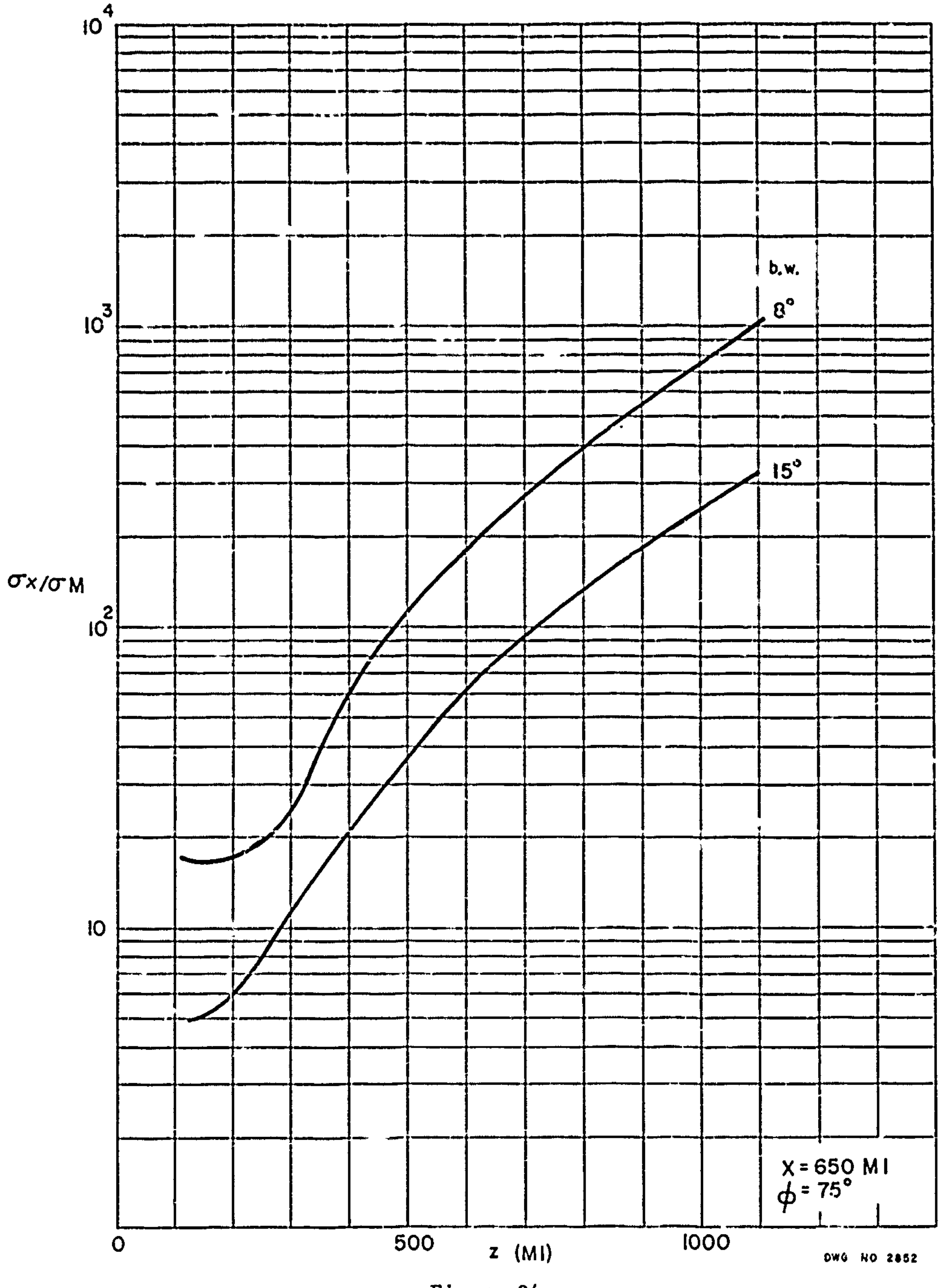

10⁴
10³
10²
10
σ_x/σ_M
b.w. = 8°
b.w = 15°
x = 350mi
ϕ = 75°
0
500
1000
z (MI)
DWG NO 2854

Figure 33a



10^3
10^2
10
σ_z/σ_M
b.w.
15°
8°
X = 350 MI
ϕ = 75°
0
500
1000
z (MI)
DWG NO 2853

Figure 33b



10^4
10^3
10^2
10
$\sigma x/\sigma M$
b.w.
8°
15°
X = 650 MI
ϕ = 75°
0
500
z (MI)
1000
DWG NO 2852

Figure 34a

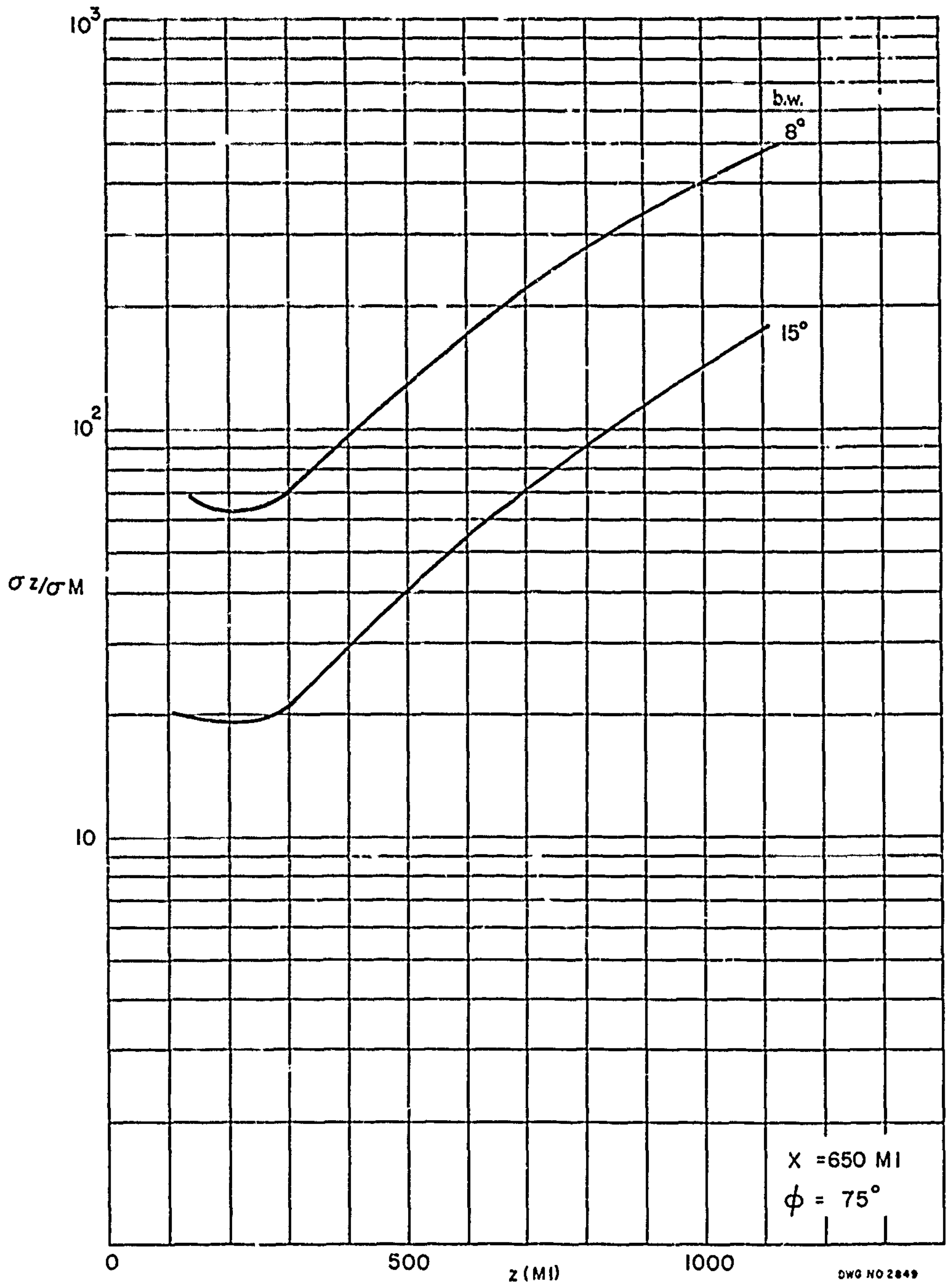
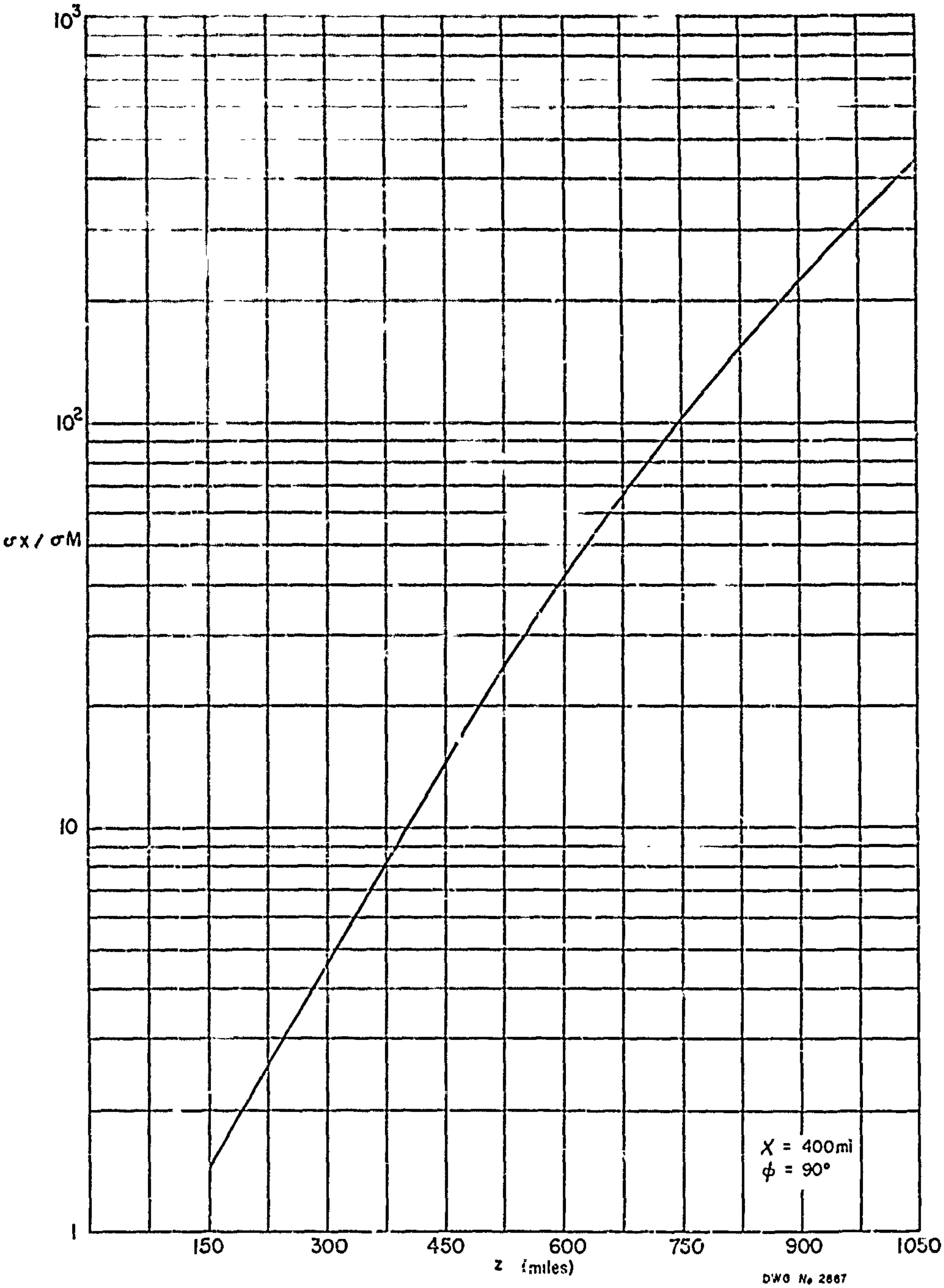


Figure 34b



10^3
10^2
$\sigma x / \sigma M$
10
1
150
300
450
600
750
900
1050
z (miles)
X = 400mi
ϕ = 90°
DWG No 2867

Figure 35a

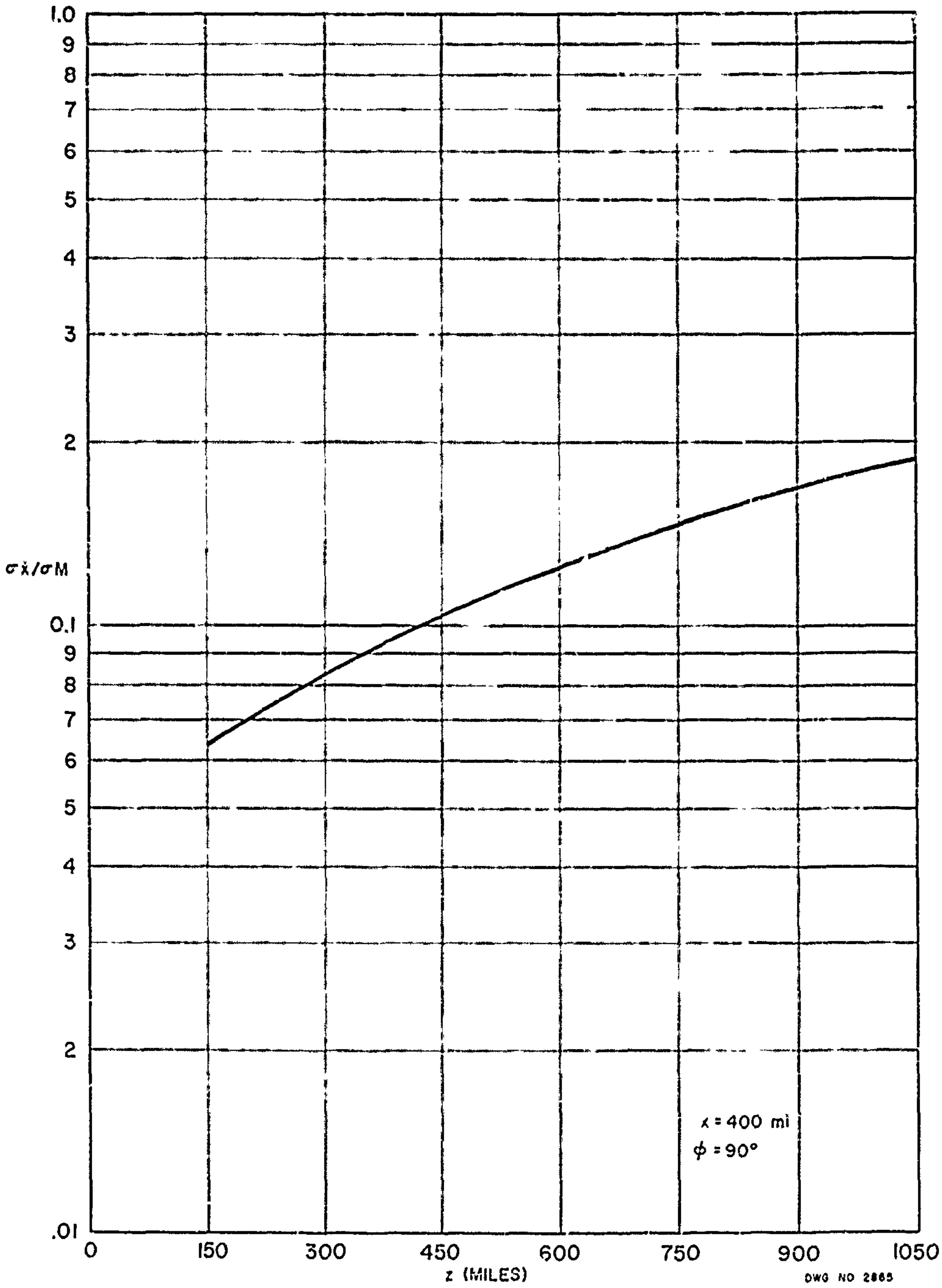
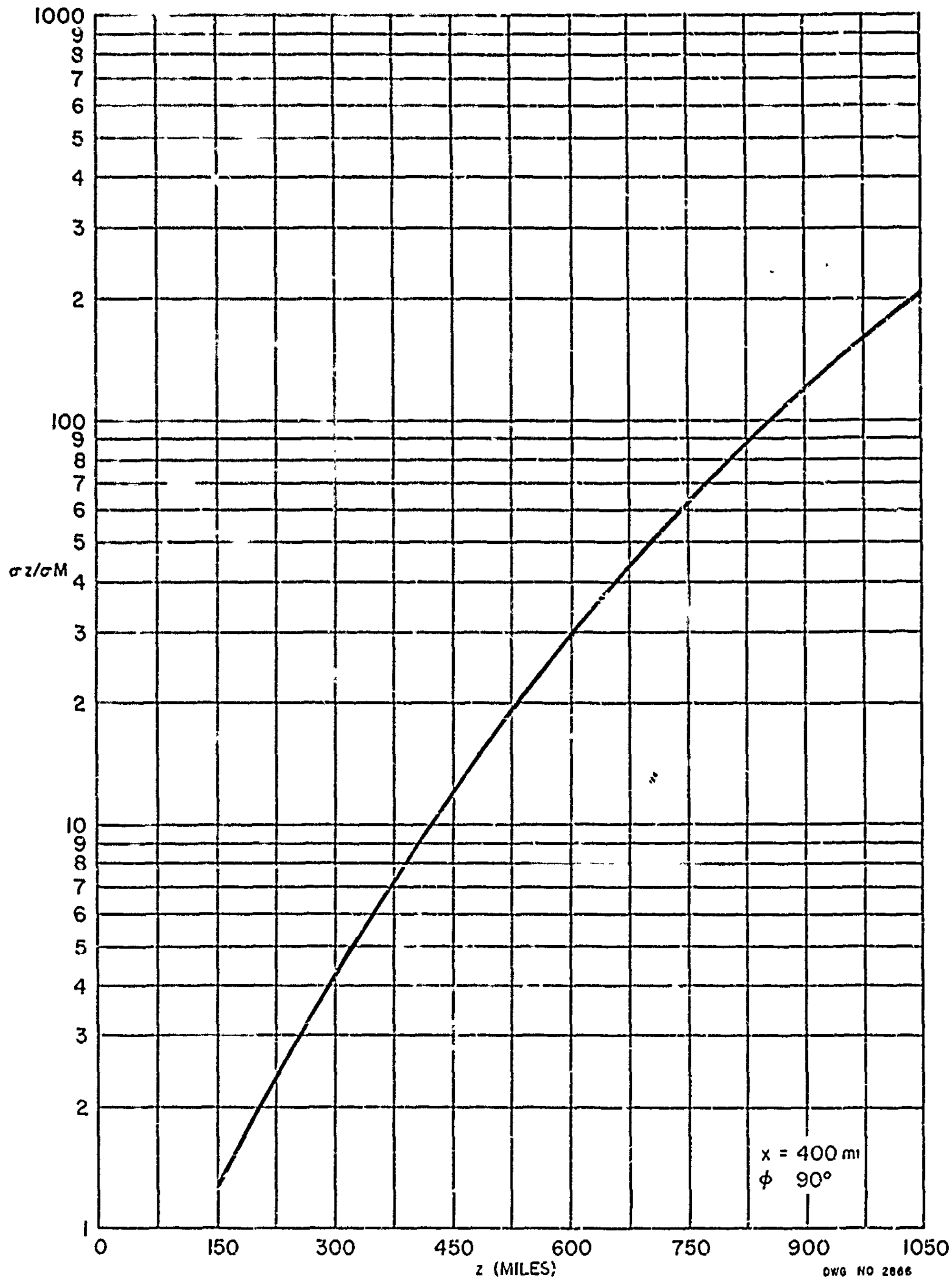

1.0
9
8
7
6
5
4
3
2
σẋ/σM
0.1
9
8
7
6
5
4
3
2
.01
x = 400 mi
φ = 90°
0
150
300
450
600
750
900
1050
z (MILES)
DWG NO 2865

Figure [illegible]b

[illegible]



1000
9
8
7
6
5
4
3
2
100
9
8
7
6
5
4
3
2
10
9
8
7
6
5
4
3
2
1
σz/σM
x = 400 mi
φ 90°
0
150
300
450
600
750
900
1050
z (MILES)
DWG NO 2866

Figure 35c

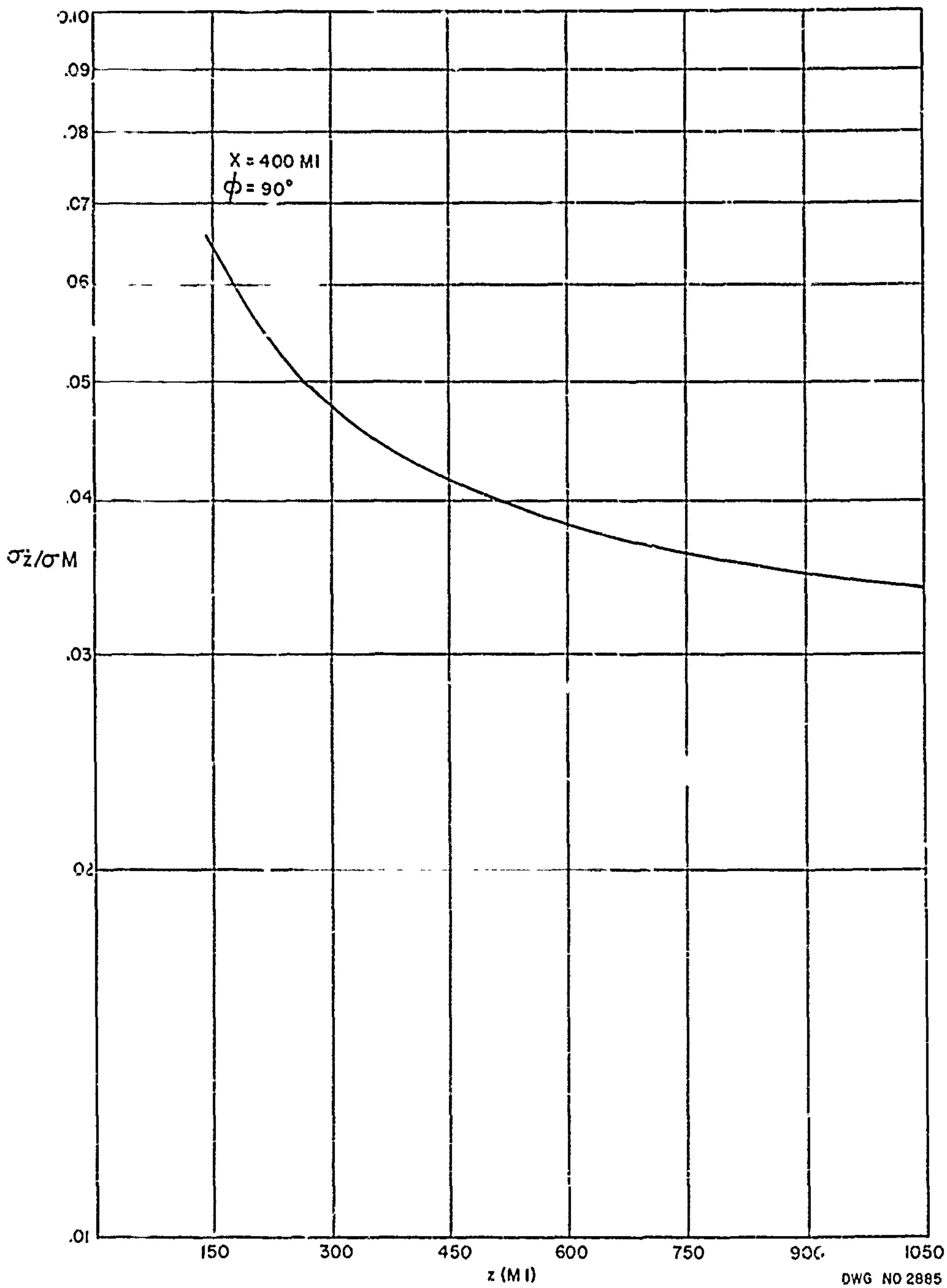

0.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
σż/σM
X = 400 MI
φ = 90°
150
300
450
600
750
900
1050
z (MI)
DWG NO 2885

Figure 35d

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